B52 Week 2 Notes

1\. - Combinations:

The number of ways to choose k objects from a total of n objects is $\frac{n!}{(k)!\,(n-k)!}$ or nCk or $\binom{c}{k}$.

- Multinominal Coefficient:

The number of ways of dividing a group of n objects into k groups of sizes $n_1, n_2, \ldots n_k$ is $\frac{n!}{(n_1)!\,(n_2)! \ldots (n_k)!}$ or $\binom{n}{n_1, n_2, \ldots n_k}$

- E.g. 1 One of my classes has 60 students. I want to divide my class into 2 groups, of even size, at random. If John and Jane are 2 students in my class, what is the prob that they will be in the same group?

Solution:

G_1 or G_2

(J, J) (J, J)

Since the prob that Jack and Jane will both be in G_1 is $\frac{\binom{58}{28}}{\binom{60}{30}}$, the prob that they will be in G_2 together is also $\frac{\binom{58}{28}}{\binom{60}{30}}$.

$\therefore$ The answer is $\frac{\binom{58}{28}}{\binom{60}{30}} \times 2$

2\. Conditional Probability:

- Let A and B be 2 events in a sample space, S. Then, the cond prob of A given B is defined by $P(A|B) = \frac{P(AB)}{P(B)}$, $P(B) > 0$.

$P(A|B)$ is read as "Probability of A given B."

- E.g. 2 If A and B are events s.t. $P(A) = 0.5$, $P(B) = 0.3$ and $P(AB) = 0.1$, find:

a) $P(A|B)$

Solution:

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3}$$

b) $P(B|A)$

Solution:

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.1}{0.5} = \frac{1}{5}$$

c) $P(A|A\cup B)$

Solution:

$$P(A|A\cup B) = \frac{P(A\cap(A\cup B))}{P(A\cup B)} = \frac{P(A)}{P(A)+P(B)-P(AB)} = \frac{0.5}{0.5+0.3-0.1} = \frac{0.5}{0.7} = \frac{5}{7}$$

d) $P(A|AB)$

Solution:

$$P(A|AB) = \frac{P(A\cap AB)}{P(AB)} = \frac{P(AB)}{P(AB)} = 1$$

Since AB is a subset of A, $A\cap AB = AB$.

- Eg. 3 A coin is flipped twice. $S = \{HH, HT, TH, TT\}$. What is the cond prob that both flips result in heads given that the first flip does?

Solution:

$A = \{HH\}$

$B = \{HH, HT\}$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{2}{4}}$$

$$= \frac{1}{2}$$

- Thm of Total Prob:

Let $A_1, A_2, \ldots A_n$ be a partition of S and let B be any event in S.

Then, $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots P(B|A_n)P(A_n)$.

Proof:

$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots P(B \cap A_n)$

However, $P(B|A_1) = \frac{P(B \cap A_1)}{P(A_1)}$

So, $P(B \cap A_1) = P(B|A_1) \cdot P(A_1)$

$\therefore P(B) = P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) \ldots + P(B|A_n) \cdot P(A_n)$

QED

- Baye's Theorem:

Let A and B be any 2 events in S, then $P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$

Proof:

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{P(B|A) \cdot P(A)}{P(B)} \text{ QED}$$

e) $P(AB \mid A \cup B)$

Solution:

$$P(AB \mid A\cup B) = \frac{P((AB)\cap(A\cup B))}{P(A\cup B)}$$

$$= \frac{P(AB)}{P(A\cup B)}$$

$$= \frac{1}{7}$$

- Thm: Conditional probability is a probability measure.
  1. $0 \le P(A \mid B) \le 1$
  2. $P(S \mid A) = 1$
  3. $P(A_1 \cup A_2 \dots A_n \mid B) = P(A_1 \mid B) + P(A_2 \mid B) \dots + P(A_n \mid B)$

Proof of 1:

$AB \subseteq B$

$P(AB) \le P(B)$

$$\frac{P(AB)}{P(B)} \le \frac{P(B)}{P(B)}$$

$$\frac{P(AB)}{P(B)} \le 1$$

$\therefore 0 \le P(A \mid B) \le 1$

QED

Proof of 2:

$$P(S \mid A) = \frac{P(S\cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)}$$

$= 1$

$= R_J$

QED

Proof of 3:

Suppose that $A_1, A_2, \dots A_n$ are disjoint events.
Then, $P(A_1 \cup A_2 \dots \cup A_n) = P(A_1) + P(A_2) + \dots P(A_n)$.

$P(A_1 \cup A_2 \dots A_n \mid B)$

$$= \frac{P((A_1 \cup A_2 \dots A_n)\cap B)}{P(B)}$$

$$= \frac{P(A_1 B \cup A_2 B \dots A_n B)}{P(B)}$$

$$= \frac{P(A_1 B) + P(A_2 B) + \dots + P(A_n B)}{P(B)}$$

$$= \frac{P(A_1 B)}{P(B)} + \dots + \frac{P(A_n B)}{P(B)}$$

$= P(A_1 \mid B) + P(A_2 \mid B) + \dots + P(A_n \mid B)$

QED

- E.g. 4 A population of voters consist of 40% Republicans and 60% Democrats. It is reported that 30% of R and 50% of D favour an election issue. A voter chosen at random favours the issue. What is the probability that this voter is a D?

Solution:

$P(D) = \frac{60}{100} = 0.6$ $\qquad$ $P(R) = \frac{40}{100} = 0.4$

$P(F|D) = \frac{50}{100} = 0.5$ $\qquad$ $P(F|R) = \frac{30}{100} = 0.3$

$$P(D|F) = \frac{P(D) \times P(F|D)}{P(F)} = \frac{(0.6)(0.5)}{(0.42)} = 0.714$$

$$P(F) = P(F|R) \cdot P(R) + P(F|D) \cdot P(D) = (0.3)(0.4) + (0.5)(0.6) = 0.42$$

3. Independent Events:

- Two events, A and B, in S are indep if $P(AB) = P(A) \times P(B)$.
  $A \perp B$ denotes that A and B are indep.

- $P(A|B) = \frac{P(AB)}{P(B)}$ $\qquad$ If $P(B) > 0$, then $A \perp B$ iff $P(A|B) = P(A)$.

$$= \frac{P(A) \times P(B)}{P(B)} \leftarrow \text{Since, if } A \perp B \text{, then } P(AB) = P(A) \times P(B)$$

$$= P(A)$$

- E.g. 5 Suppose we roll a fair, six-sided die.

$A = \{1,3,5\}$

$B = \{2,4,6\}$

$C = \{1,2\}$

$D = \{1,2,3,4,5,6\}$

a) Are A and B indep?

Solution:

$P(AB) = 0$

$\neq P(A) \times P(B)$

$\therefore$ Not indep

b) Are A and C indep?

Solution:

$P(A) = \frac{3}{6} = \frac{1}{2}$

$P(C) = \frac{2}{6} = \frac{1}{3}$

$P(AC) = \frac{1}{6}$

$= \frac{1}{2} \times \frac{1}{3}$

$= P(A) \times P(C)$

$\therefore A \perp C$

- Note: If A and B are any 2 elements in S s.t. $P(A) > 0$ and $P(B) > 0$, if A and B are disjoint, then A and B are not indep.

Proof:

If A and B are disjoint,

$P(AB) = 0$

$\neq P(A) \times P(B)$

$\therefore$ A and B are not indep.

QED

- Thm: $A \perp B \Rightarrow A \perp B^c$

Proof:

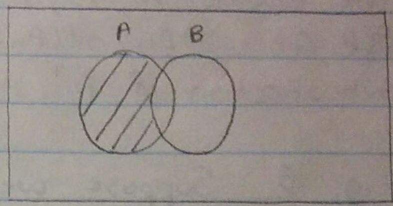
Suppose A and B are indep.

That means, $P(AB) = P(A) \cdot P(B)$

$$
\begin{aligned}
P(A \cap B^c) &= P(A \setminus B) \\
&= P(A) - P(AB) \\
&= P(A) - P(A) \cdot P(B) \\
&= P(A)[1 - P(B)] \\
&= P(A) \cdot P(B^c), \text{ as wanted}
\end{aligned}
$$

I.e. $A \perp B^c$

QED



The shaded region is $P(A \cap B^c)$.

- Thm: $A \perp B \Rightarrow A^c \perp B^c$

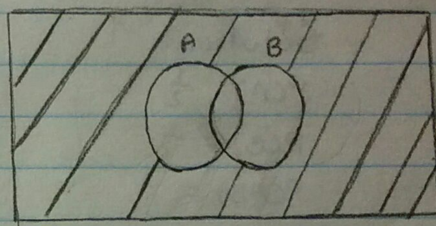
Proof:

Suppose A and B are indep.

That means $P(AB) = P(A) \cdot P(B)$

$$
\begin{aligned}
P(A^c \cap B^c) &= P(S) - P(A \cup B) \\
&= 1 - P(A \cup B) \\
&= 1 - [P(A) + P(B) - P(AB)] \\
&= 1 - P(A) - P(B) + P(AB) \\
&= (1 - P(A)) - P(B) + P(A) \cdot P(B) \\
&= P(A^c) - P(B) + P(A) \cdot P(B) \\
&= -[-P(A^c) + P(B) - P(A) \cdot P(B)] \\
&= -[-P(A^c) + P(B)[1 - P(A)]] \\
&= -[-P(A^c) + P(B) \cdot P(A^c)] \\
&= P(A^c) - P(B) \cdot P(A^c) \\
&= P(A^c)[1 - P(B)] \\
&= P(A^c) \cdot P(B^c), \text{ as wanted}
\end{aligned}
$$

QED



The shaded area is $A^c \cap B^c$ because $A^c \cap B^c$ is the area not in A and not in B.

- The events $A_1, A_2, A_3, \dots A_n$ in S are indep if $P(A_{i_1} \cap A_{i_2} \dots \cap A_{i_k}) = P(A_{i_1}) \times P(A_{i_2}) \times P(A_{i_k})$ for each finite subcollection of $A_{i_1}, A_{i_2}, \dots A_{i_k}$.

- E.g. 6 Suppose we roll a 4-sided die.

$S = \{1, 2, 3, 4\}$

$A = \{1, 2\}$

$B = \{1, 3\}$

$C = \{1, 4\}$

a) Are A and B indep?

Solution:

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

$P(AB) = \frac{1}{4}$

$= \frac{1}{2} \times \frac{1}{2}$

$= P(A) \cdot P(B)$

$\therefore$ Yes, $A \perp B$

b) Are B and C indep?

Solution:

$P(B) = \frac{1}{2}$

$P(C) = \frac{1}{2}$

$P(B \cap C) = \frac{1}{4}$

$= \frac{1}{2} \times \frac{1}{2}$

$= P(B) \cdot P(C)$

$\therefore$ Yes, $B \perp C$

c) Are A and C indep?

Solution:

$P(A) = \frac{1}{2}$

$P(C) = \frac{1}{2}$

$P(AC) = \frac{1}{4}$

$= \frac{1}{2} \times \frac{1}{2}$

$= P(A) \cdot P(C)$

$\therefore$ Yes, $A \perp C$

d) Are A and B and C indep?

Solution.

$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{4}$ (Because $P(A \cap B \cap C) = \{1\}$)

$\neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$\therefore$ A and B and C are not indep.

Note: Pair-wise indep does not imply indep.