B52 Week 1 Notes

1. Relative Frequency of Probability
Let A be an event in a random experiment.
Let $r_n(A)$ be the number of times A occurs in n repeated trials of the experiment.
Let $P(A)$ be the probability that A occurs.

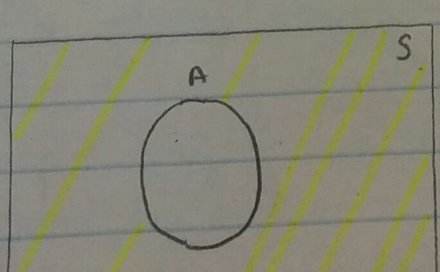
$$P(A) = \lim_{n \to \infty} \left(\frac{r_n(A)}{n} \right)$$

Frequency is another term for frequency.

2. Formal Definition of Probability
1. Sample Space: The set of all possible outcomes in an experiment. It is denoted by S.
E.g. If the experiment is tossing a coin, then the sample space, $S = \{H, T\}$.

2. Event: It is a subset of a sample space.
Note: The null set or empty set is denoted by $\emptyset$.
It is a subset of S, so it is an event.
E.g. Continuing with our coin experiment,
- If Event A is the event that heads is face up after the toss, then Event $A = \{H\}$.

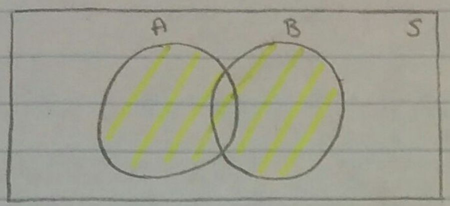
3. Complement:
Suppose A is an event. Then A^c is the complement to that event. A^c is the set of elements in S that's not in A.



The shaded portion of S represents A^c.

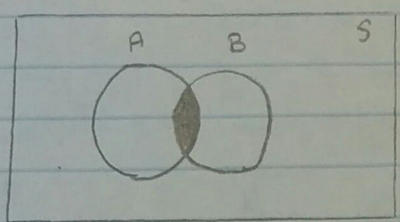
4. Union and Intersection

- $\cup$ is the symbol for Union.
  $A \cup B$ is the set of elements in A or B or both.



The shaded area represents $A \cup B$.

- $\cap$ is the symbol for intersection.
  $A \cap B$ is the set of elements in both A and B.



The shaded area represents $A \cap B$.
Note: $A \cap B$ can also be represented as AB.

E.g. Let $A = \{1, 2, 3\}$
Let $B = \{3, 4\}$
$A \cup B = \{1, 2, 3, 4\}$
$A \cap B = \{3\}$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$
- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$

5. Formal Definition of Probability

Let S be a sample space. A function, P, defined on the events of S is called a probability measure if it satisifies these conditions:

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. Let $A_1, A_2, A_3, \ldots A_n$ be a sequence of pairwise disjoint events. I.e. $A_1 \cap A_2 = \emptyset$
   Then, $P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots P(A_n)$
4. $P(\emptyset) = 0$

3. Theorems:

1. $P(A_1 \cup A_2 \ldots \cup A_n) = \sum_{i=1}^{n} A_i$ if $A_i \cap A_j = \emptyset \ \forall i \neq j$

Proof:

Define A_i to be the null set $\forall i > n$

Then $A_i \cap A_j = \emptyset \ \forall i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i$ Note: $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$

$$P\left(\bigcup_{i=1}^{n}\right) = P\left(\bigcup_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i)$$

$= 0$ (Because $A_i = \emptyset \ \forall i > n$, and $P(\emptyset) = 0$)

$$= \sum_{i=1}^{n} P(A_i)$$

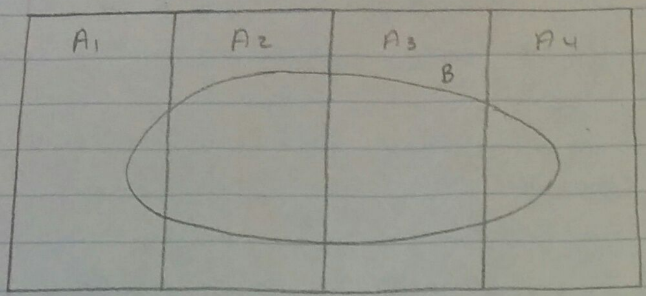
QED

2. Thm of Total Probably

Let $A_1, A_2, A_3 \ldots, A_n$ be a partition of S.

If $A_i \cap A_j = \emptyset \ \forall i = j$ and $\bigcup_{i=1}^{\infty} A_i = S$ and B is in any event in S, then $P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$

Proof:



From the diagram, we can see that $(B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup (B \cap A_4)$
$= B$.

$\therefore \quad B = \bigcup_{i=1}^{\infty} (B \cap A_i)$

Furthermore, we know that $B \cap A_i$'s are pairwise disjoint.

I.e. $(B \cap A_i) \cap (B \cap A_j) = \emptyset \quad \forall i \neq j$

$$\therefore \quad P(B) = P\left(\bigcup_{i=1}^{\infty} (B \cap A_i)\right)$$
$$= \sum_{i=1}^{\infty} P(B \cap A_i)$$

QED

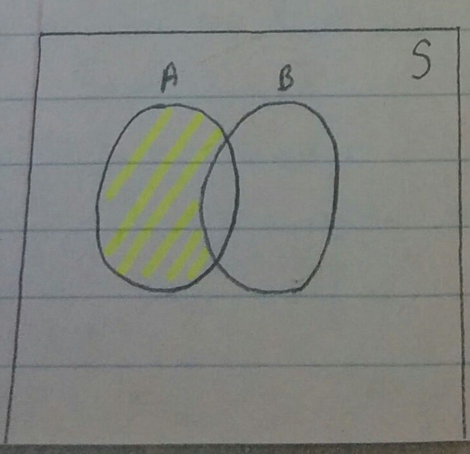
3.

Let A and B be 2 events in S.

$A \setminus B = A \cap B^c$

$\quad = $ Anything in A and not in B

$A \setminus B$ is read as A minus B



The shaded area represents $A \setminus B$.

$P(A \setminus B) = P(A) - P(A \cap B)$

Proof:

Note that:

1. $A = (A \setminus B) \cup (A \cap B)$
2. $(A \setminus B)$ and $(A \cap B)$ are disjoint

} Can be seen on the diagram

Since $A = (A \setminus B) \cup (A \cap B)$,

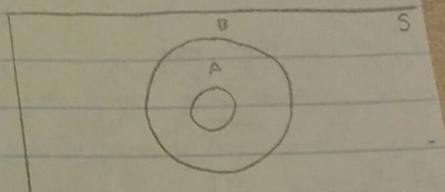
$P(A) = P((A \setminus B) \cup (A \cap B))$

$= P(A \setminus B) + P(A \cap B)$ because they are disjoint

$P(A \setminus B) = P(A) - P(A \cap B)$

QED

4. If $A \subseteq B$, then $P(A) \leq P(B)$



$P(A \cap B) = P(A)$ in this case.

Proof:

$P(B \setminus A) = P(B) - P(A \cap B)$ (From last thm)

$= P(B) - P(A)$

We know that $0 \leq P(H) \leq 1$

$\therefore$ $P(B) - P(A) \geq 0$

$P(A) \leq P(B)$

QED

5. $P(A^c) = 1 - P(A)$

Proof:

$A^c = S \setminus A$

$P(A^c) = P(S \setminus A)$

$= P(S) - P(S \cap A)$

$= 1 - P(A)$ (By prev axioms)

QED

6. If A and B are any 2 events in a sample space, S, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

$A \cup B = A \cup (B \setminus A)$

Since A and $B \setminus A$ are disjoint, $P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$

$$\begin{aligned} P(A \cup B) &= P(A \cup (B \setminus A)) \\ &= P(A) + P(B \setminus A) \\ &= P(A) + P(B) - P(A) \end{aligned}$$

QED

Note: $P(A \cup B) \leq P(A) + P(B)$

For any collection of events, $A_1, A_2, \ldots A_n$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{n} A_i$$

7. For any 3 events, A, B, C, in sample space, S,
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Proof:

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup (B \cup C)) \\ &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B) + P(C) - P(BC) - [P(A \cap B) \cup P(B \cap C)] \\ &= P(A) + P(B) + P(C) - P(BC) - [P(AB) + P(BC) - P(ABC)] \\ &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(BC) + P(ABC) \end{aligned}$$

QED

E.x. There are 33 students in the class. 17 of them earned an A on the midterm. 14 of them earned an A on the final exam. 11 of them did not earn an A on either test. ① How many students got an A on both the midterm and the final? ② What is the probability that a randomly selected student earned an A on both tests?

Since there are 33 students and 11 of them didn't get an A on either test, 33-11 or 22 students got at least an A. $17+14=31$. $31-22=9$.

$\therefore$ 9 students got an A on both tests.

$\therefore$ The probability is $\frac{9}{33}$.

4. Uniform Probability Measure on Finite Sample Spaces:

Let S be a sample space with equally likely outcomes and has a finite number of outcomes. Let A be any event in S. The uniform probability measure is given by: $P(A) = \frac{|A|}{|S|}$

I.e. $P(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } S}$

5. Combinatorics:

- Permutation occurs when the order matters.
  The formula for permutation is: $\frac{n!}{(n-r)!}$

where n is the number of items to choose from and we choose r of those items without reptition.

Ways of writing permutations:

nPr, $P(n,r)$

- Combinations occur when the order doesn't matter. The formula for combinations is $\frac{n!}{r!(n-r)!}$ or $\frac{P(n,r)}{r!}$

where n is the number of items to choose from and we choose r of those items without reptition. Another term for combinations is the binomial coefficient.

Ways of Writing Combinations:
nCr, $C(n,r)$, $\binom{c}{r}$

- Multinomial Coefficient is used permutations when you have duplicate items.
Its formula is $\frac{n!}{k_1!\,k_2!\ldots k_n!}$ where n is the number of items to choose from and $k_1, k_2, \ldots k_n$ are the number of each repeated item.

E.g. Find the number of different permutations for the word 'MISSISSIPPI.'

Note that:
1. There are 11 letters, so $n = 11$.
2. There is 1 M. $k_1 = 1$ (Since $1! = 1$, you don't have to show this)
3. There are 4 I's. $k_2 = 4$
4. There are 4 S's. $k_3 = 4$
5. There are 2 P's. $k_4 = 2$

$$\frac{n!}{k_1!\,k_2!\,k_3!\,k_4!}$$

$$= \frac{11!}{(1!)(4!)(4!)(2!)}$$

$= 34,650$ different number of permutations.

6. Multiplication Principle:

If one event can occur in M ways and another event, independent of the first, can occur in N ways, then the two events can occur in $M \cdot N$ ways.

E.g. Suppose we flip 3 fair coins and roll two fair six-sided die. What is the probability that all 3 coins land heads up and both dice come up 6?

Solution:

Since flipping 3 coins and rolling 2 dice can be performed independent of the other, we will use the multiplication principle.

Probably of flipping 3 heads:

$(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{8}$

Probability of rolling 2 sixes:

$(\frac{1}{6})(\frac{1}{6}) = \frac{1}{36}$

$$\text{Total Probability} = \frac{1}{8} \cdot \frac{1}{36}$$
$$= \frac{1}{288}$$

Note: If the question has the word and, like in our question, use the multiplication principle.

Another term for 'Multiplication Principle' is 'Rule of Product.'

7. Rule of Sum:

- If an event can occur M ways and another event can occur N ways and the two events cannot be done at the same time, there are M+N ways to choose one of the ways.

E.g. Suppose there are 10 balls in a bag. Of the 10 balls, 3 are red, 4 are yellow, 2 are blue and 1 is green. If you reach into the bag without looking and randomly pick a ball, what is the probability that you chose a red or blue ball?

Solution:

Probability of picking a red ball: $\frac{3}{10}$

Probability of picking a blue ball: $\frac{2}{10}$

Prob. of picking a red or blue ball $= \frac{3}{10} + \frac{2}{10}$

$= \frac{5}{10}$

$= \frac{1}{2}$

- Sometimes there will be overlap between the events. If there are any overlaps, just subtract the number of ways that the overlap can occur from M+N.

$P(M \cup N) = P(M) + P(N) - P(M \cap N)$ is used to calculate the Rule of Sum.

$P(M \cup N)$ is the total number of ways to choose one of the ways.

$P(M)$ is the number of ways event M can occur.

$P(N)$ is the number of ways event N can occur.

$P(M \cap N)$ is the number of ways event M and event N overlaps. If event M is completely independent of event N, $P(M \cap N) = 0$.

Note: If the question has or, use the Rule of Sum.

Note: Another term for 'Rule of Sum' is 'Addition Principle.'

8. Probability of getting k heads in n tosses:

Suppose you want to calculate the probability of getting k heads in n flips of a fair coin. Then: $P(k) = \frac{n!}{(k!)(n-k)!(2^n)}$

E.g. Suppose you want to calculate the probability of getting exactly 3 heads in 10 flips of a fair coin.

Solution:

$n = 10$

$k = 3$

$$P(k) = \frac{n!}{(k!)(n-k)!(2^n)}$$

$$= \frac{10!}{(3!)(10-3)!(2^{10})}$$

$$= \frac{10!}{(3!)(7!)(2^{10})}$$

$$= \frac{10 \times 9 \times 8}{(3!)(2^{10})}$$

$$= \frac{10 \times 3 \times 4}{2^{10}}$$ (Because $3! = 6$ and $72 \div 6 = 12$)

$$= \frac{120}{2^{10}}$$