PMF, PDF, CDF

1. Probability Mass Function (PMF):

Denoted by $P_X(x)$ or $P(X=k)$

Properties of PMF:

1. $0 \leq P(X=k) \leq 1 \quad \forall k$
2. $\sum P(X=k) = 1$

E.g. Write the PMF of rolling a die.

$P_X(1) = P(X=1) = \frac{1}{6}$

$P_X(2) = P(X=2) = \frac{1}{6}$

$P_X(3) = P(X=3) = \frac{1}{6}$

$P_X(4) = P(X=4) = \frac{1}{6}$

$P_X(5) = P(X=5) = \frac{1}{6}$

$P_X(6) = P(X=6) = \frac{1}{6}$

$P_X(x) = P(X=x) = 0 \leftarrow$ This means that if x is not 1, 2, 3, 4, 5 or 6, then $P_X(x) = 0$.

Another way to show PMF:

$$P_X(x) = \begin{cases} \frac{1}{6}, & x = 1, 2, 3, 4, 5, \text{ or } 6 \\ 0, & \text{otherwise} \end{cases}$$

PMF gives you the probability for discrete random variables.

A random variable, k, is discrete if there is a finite sequence of real numbers, $k_1, k_2, \ldots$ and a corresponding sequence of positive real numbers $P_1, P_2, \ldots$ such that 1. $P(X=k_i) = P_i$

2. $\sum_i P_i = 1$

2. Probability Density Function (PDF)

Denoted by $f_X(x)$.

Properties:

1. $f_X(x) \geq 0 \ \forall x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

PDF gives you the probability for continuous random variables.

A random variable, k, is continuous if $P(X=k) = 0 \ \forall k \in \mathbb{R}$.

The PDF is the derivative of the CDF.

E.g. A r.v. has pdf given by:

$$f_X(x) = \begin{cases} kx, & \text{if } 0 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

Find k.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\underbrace{\int_{-\infty}^{0} f_X(x)\, dx}_{0} + \int_{0}^{5} f_X(x)\, dx + \underbrace{\int_{5}^{\infty} f_X(x)\, dx}_{0} = 1$$

$$\int_0^5 kx\, dx = 1$$

$$k\int_0^5 x\, dx = 1$$

$$k\left[\frac{x^2}{2}\Big|_0^5\right] = 1$$

$$\Rightarrow \frac{25k}{2} = 1$$

$$k = \frac{2}{25}$$

Absolutely Continuous Random Variables:

A r.v., X, is abs cont if there exists a density function s.t. $P(a \le X \le b) = \int_a^b f_X(x)\,dx$, whenever $a \le b$.

Note: Abs cont ran vars are always cont, but the converse isn't true.

E.g.

$$f_X(x) = \begin{cases} \frac{2x}{25}, & \text{if } 0 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(2 \le X \le 3)$

$$\int_2^3 f_X(x)\,dx$$

$$= \int_2^3 \frac{2x}{25}\,dx$$

$$= \frac{2}{25}\int_2^3 x\,dx$$

$$= \frac{2}{25}\left[\frac{x^2}{2}\Big|_2^3\right]$$

$$= \frac{1}{25}[9-4]$$

$$= \frac{1}{5}$$

Abs Cont Distributions:

1. Uniform Distribution over $[L, R]$:

A r.v. X has a uniform dist over $[L, R]$ if

$$f_X(x) = \begin{cases} \frac{1}{R-L}, & L \leq x \leq R \\ 0, & \text{otherwise} \end{cases}$$

2. Exponental Distribution:

A r.v. X has an exp dist with parameter λ if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \text{ and } \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

This is denoted by $X \sim \exp(\lambda)$.

E.g. $X \sim \exp(0.2)$
Find $P(X > 10)$

$$\int_{10}^{\infty} f_X(x)\,dx$$

$$= \int_{10}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \int_{10}^{\infty} 0.2 e^{-0.2x}\,dx$$

$$= \left[-e^{-0.2x} \Big|_{10}^{\infty} \right]$$

$$= 0 - (-e^{-(0.2)(10)})$$

$$= e^{-2}$$

3. Normal Distribution:

A r.v. x has a normal dist with parameters μ and σ^2, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$ if

$$f_X(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, & x \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

It is denoted as $X \sim N(\mu, \sigma^2)$.

Note: $X \sim N(0,1)$ is called the standard normal dist

and $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2)}, \quad x \in \mathbb{R}$

4. Gamma Distribution:

A r.v. x has a Gamma dist with parameters α and λ ($\alpha > 0$, $\lambda > 0$) if

$$f_X(x) = \begin{cases} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x > 0,\ \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim \Gamma(\alpha, \lambda)$

Properties:

1. $\Gamma(1) = 1$
2. If $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
3. If α is a positive int, $\Gamma(\alpha) = (\alpha-1)!$
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\int_0^\infty \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\, dx = 1$$

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

5. Beta Distribution:

A r.v. x has a beta dist with parameters a and b, $a>0$, $b>0$ if

$$f_X(x) = \begin{cases} \dfrac{\Gamma(a+b)\cdot x^{a-1}\cdot(1-x)^{b-1}}{\Gamma(a)\,\Gamma(b)}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim Beta(a,b)$

$$\int_0^1 \frac{\Gamma(a+b)\cdot x^{a-1}\cdot(1-x)^{b-1}}{\Gamma(a)\cdot\Gamma(b)}\, dx = 1$$

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, dx = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

3. Cumulative Distribution Function (CDF)

Denoted by $F_X(x)$, or $P(X \le x)$

Properties:

1. $0 \le F_X(x) \le 1 \quad \forall x \in \mathbb{R}$
2. $a \le b \rightarrow F_X(a) \le F_X(b)$
3. $\lim_{x \to \infty} F_X(x) = 1$
4. $\lim_{x \to -\infty} F_X(x) = 0$
5. $P(a < X \le b) = F_X(b) - F_X(a)$
6. $P(a \le X \le b) = F_X(b) - F_X(a^-)$
7. $P(a < X < b) = F_X(b^-) - F_X(a^-)$
8. $P(a \le X < b) = F_X(b^-) - F_X(a^-)$

Note: $F_X(a^-) = P(X < a)$, $F_X(b^-) = P(X < b)$

$$F_X(a^-) = \lim_{n \to \infty} F_X(a - \tfrac{1}{n})$$
$$= P(X < a)$$
$$F_X(b^-) = \lim_{n \to \infty} F_X(b - \tfrac{1}{n})$$
$$= P(X < b)$$

Note: 1. $P(X = a) = P(X \le a) - P(X < a)$
$$= F_X(a) - F_X(a^-)$$

2. $P(X \le a) = P(X = a) + P(X < a)$

3. $P(X < a) = P(X \le a) - P(X < a)$

Note: $f_X(x) = \dfrac{d F_X(x)}{dx}$

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt$$