One Dimensional Change of Variables

Given a r.v. X, suppose we know $P_X(x)$ and $f_X(x)$ if $Y=h(X)$, but we want to find $P_Y(y)$ or $f_Y(y)$.

E.g. Suppose that X has PMF

$$P_X(x) = \begin{cases} \frac{1}{7}, & x \in \{-3,-2,-1,0,1,2,3\} \\ 0, & \text{otherwise} \end{cases}$$

Find the PMF of $Y = X^2 - X$.

1. $P_Y(0) = P(X=0) + P(X=1)$
   $= 2/7$

2. $P_Y(2) = P(X=2) + P(X=-1)$
   $= 2/7$

3. $P_Y(6) = P(X=-2) + P(X=3)$
   $= 2/7$

4. $P_Y(12) = P(X=-3)$
   $= \frac{1}{7}$

$$P_Y(y) = \begin{cases} \frac{2}{7}, & \text{if } y \in \{0,2,6\} \\ \frac{1}{7}, & \text{if } y \in \{12\} \\ 0, & \text{otherwise} \end{cases}$$

This is the discrete case.

Abs Cont Case:

1. Use the CDF to find PDF.

E.g. Let x be a r.v. with PDF

$$f_x(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of $y = 2x - 1$

$$\begin{aligned} F_y(y) &= P(y \le y) \\ &= P(2x-1 \le y) \\ &= P(x \le \tfrac{y+1}{2}) \\ &= F_x(\tfrac{y+1}{2}) \end{aligned}$$

$$\begin{aligned} f_y(y) &= \frac{d}{dx} F_y(y) \\ &= \frac{d}{dx} F_x(\tfrac{y+1}{2}) \\ &= \frac{1}{2} f_x(\tfrac{y+1}{2}) \\ &= \begin{cases} \cancel{2}(1-\frac{y+1}{2})(\cancel{\tfrac{1}{2}}), & 0 \le \frac{y+1}{2} \le 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} (\frac{1-y}{2}), & -1 \le y \le 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

2. If x is an abs cdf r.v. and $y = h(x)$ where $h: \mathbb{R} \to \mathbb{R}$ is differentiable, and strictly inc or dec on the support of $f(x)$, then y is an abs cdf and $f_y(y) = \dfrac{f_x(h^{-1}(y))}{|h'(h^{-1}(y))|}$ where $h'(x) = \frac{d}{dx} h(x)$.

The support of $f(x)$ means the interval for which $f(x)$ is positive.

Using the previous example:

$h(x) = 2(1-x)$

$\therefore$ $h(x)$ is strictly decreasing.

$$f_y(y) = \frac{f_x(h^{-1}(y))}{|h'(h^{-1}(y))|}$$

Let $y = h(x) = 2(1-x)$.

$h^{-1}(y) = \frac{y+1}{2}$

$h'(x) = \frac{d}{dx} h(x)$

$= \frac{d}{dx}(2x-1)$

$= 2$

$$= \frac{f_x\left(\frac{y+1}{2}\right)}{2}$$

$$= \frac{2\left(1 - \frac{y+1}{2}\right)}{2}$$

$$= \frac{1-y}{2}$$

$$f_y(y) = \begin{cases} \frac{1-y}{2}, & -1 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$