Joint and Marginal Distributions

1. Definition:

Let x and y be 2 r.v.. The set of all $P((x,y) \in B)$, $\forall B \subseteq R^2$ is the joint dist of x and y.

2. Joint CDF:

Let x and y be 2 r.v.. The joint CDF of x and y is the function $F_{x,y}(x,y) : R^2 \to [0,1]$ defined by $F_{x,y}(x,y) = P(X \le x, Y \le y)$. The comma means "and,"
I.e. $F_{x,y}(x,y) = P(X \le x \cap Y \le y) = P(X \le x, Y \le y)$.

3. Marginal Dist for Joint CDF:

1. $F_x(x) = \lim_{y \to \infty} F_{x,y}(x,y)$

2. $F_y(y) = \lim_{x \to \infty} F_{x,y}(x,y)$

Note:
1. $F_{x,y}(x \le \infty, y \le \infty) = 1$
2. $F_{x,y}(x \le -\infty, y \le -\infty) = 0$
3. $F_{x,y}(x \le -\infty, y \le y) = 0$
4. $F_{x,y}(x \le x, y \le -\infty) = 0$

4. Joint PMF:

Let x and y be 2 discrete r.v. The joint PMF of x and y is the function $P_{x,y} : R^2 \to [0,1]$ be defined by $P_{x,y}(x,y) = P(X = x, Y = y)$.

5. Marginal Dist for Joint PMF:

1. $P_X(x) = \sum_y P_{X,Y}(x,y)$

2. $P_Y(y) = \sum_x P_{X,Y}(x,y)$

6. Joint PDF:

A function, f, is a joint PDF if it satisfies

1. $f_{X,Y}(x,y) \geq 0 \ \forall (x,y) \in \mathbb{R}^2$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = 1$

Note: (X,Y) is absolutely continuous if there exists a function, f, s.t. $P(a \leq X \leq b, c \leq Y \leq d)$
$= \int_c^d \int_a^b f_{X,Y}(x,y) \, dx \, dy.$

7. Marginal Dist for Joint PDF:

1. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy$

2. $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx$

E.g. 1 Given this chart

y \ x	1	2	3	4	5	6
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

a) Find $F_{x,y}(2,0)$

$$F_{x,y}(2,0) = P(x \le 2, y \le 0) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

b) Find the marginal joint pmf of x.

$$P_x(x) = \sum_y P_{x,y}(x,y)$$

y \ x	1	2	3	4	5	6	$P_y(y)$
0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
$P_x(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	

x	$P_x(x)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

← Marginal Joint PMF of x

c) Find the marginal joint pmf of y

y	$P_y(y)$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

← Marginal Joint PMF of y

E.g. 2 $f_{x,y}(x,y) = \begin{cases} kxy, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

a) Find the value of k that makes this a pdf.

$$1 = \int_0^1 \int_0^1 kxy \, dx\, dy$$

$$= k \int_0^1 \int_0^1 xy \, dx\, dy$$

$$= k \int_0^1 y(1-0) \left[\frac{x^2}{2} \Big|_0^1 \right] dy$$

$$= k \int_0^1 \frac{y}{2} \, dy$$

$$= \frac{k}{2} \left[\frac{y^2}{2} \Big|_0^1 \right]$$

$$= \frac{k}{4}$$

$$k = 4$$

b) Find $P(x \le \frac{1}{2}, y \le \frac{3}{4})$

$$P(x \le \tfrac{1}{2}, y \le \tfrac{3}{4}) = 4 \int_0^{3/4} \int_0^{1/2} xy \, dx\, dy$$

$$= 4 \int_0^{3/4} y(\tfrac{1}{2}-0) \left[\frac{x^2}{2} \Big|_0^{\frac{1}{2}} \right] dy$$

$$= \frac{4}{16} \int_0^{3/4} y \, dy$$

$$= \frac{1}{4} \left[\frac{y^2}{2} \Big|_0^{3/4} \right]$$

$$= \frac{1}{8} \left[\frac{9}{16} \right]$$

$$= \frac{9}{128}$$