Inequalities

1. Markov's Inequality:

If X is a non-negative r.v. then for all $a>0$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

E.g. Let $X \sim Exp(3)$. Find Markov's upper band for $P(X \geq 5/6)$.

$$P(X \geq 5/6) \leq \frac{E(X)}{a}$$

$$\leq \frac{\frac{1}{3}}{5/6}$$

$$\leq 2/5$$

2. Chebyshev's Inequality:

Let X be an arbitrary r.v. with finite mean $E(X)$. Then, for all $a>0$, $P(|X-E(X)| \geq a) \leq \frac{Var(X)}{a^2}$

Proof:
Let $Y = (X-E(X))^2$. Then, Y is a non-negative r.v. and we can use Markov's Inequality.

$$P(|X-E(X)| \geq a) \rightarrow P(Y \geq a^2)$$

$$P(Y \geq a^2) \leq \frac{E(Y)}{a^2}$$

$$\leq \frac{E(X-E(X))^2}{a^2}$$

$$\leq \frac{Var(X)}{a^2}$$

E.g. Let $X \sim \text{Exp}(3)$. Find Chebyshev's bound for $P(|X - \frac{1}{3}| \geq \frac{1}{2})$.

$$P(|X - \tfrac{1}{3}| \geq \tfrac{1}{2}) \leq \frac{\text{Var}(X)}{a^2}$$
$$\leq \frac{(\frac{1}{3})^2}{(\frac{1}{2})^2}$$
$$\leq 4/9$$

3. Cauchy-Schwartz:

$$(E(XY))^2 \leq E(X^2) \cdot E(Y^2)$$
$$(\text{Cov}(X,Y))^2 \leq \text{Var}(X) \cdot \text{Var}(Y)$$
$$|\text{Corr}(X,Y)| \leq 1$$