Expectations

1. The Discrete Case:

- If x is a discrete rv with pmf $P_x(x)$, then the expectation of x is defined by $E(x) = \sum_x x P_x(x)$.

Eg. 1. A r.v. x has pmf given in this table.

x	$P_x(x)$
0	0.2
1	0.3
2	0.5

Find $E(x)$.

$$E(x) = \sum_x x P_x(x)$$

$$= (0)(0.2) + (1)(0.3) + (2)(0.5)$$

$$= 1.3$$

Note, $E(x)$ could be negative.

- Expectations of Some Discrete Distributions:

1. $x \sim \text{Degenerate}(c) \rightarrow E(x) = c$
2. $x \sim \text{Bernoulli}(\theta) \rightarrow E(x) = \theta$
3. $x \sim \text{Bin}(n, \theta) \rightarrow E(x) = n\theta$
4. $x \sim \text{Geo}(\theta) \rightarrow E(x) = \frac{1-\theta}{\theta}$
5. $x \sim \text{Po}(\lambda) \rightarrow E(x) = \lambda$
6. $x \sim \text{Neg-Bin}(r, \theta) \rightarrow E(x) = \frac{r(1-\theta)}{\theta}$
7. $E(c) = c$, where c is a constant.

- Let X be a discrete r.v. and let $g: R^1 \to R^1$ be some function s.t. the expectation of the r.v. $g(X)$ exists. Then:

$$E(g(X)) = \sum_{x} g(x) P(X=x)$$

E.g. 2. Given this table,

x	$P_X(x)$
1	0.5
2	0.5

find $E(\frac{1}{X})$.

Let $g(x) = \frac{1}{x}$

$$E(g(X)) = \sum_{x} g(x) P_X(x)$$

$$= 1(0.5) + \frac{1}{2}(0.5)$$

$$= \frac{3}{4}$$

Note: $E(\frac{1}{X}) \neq \dfrac{1}{E(X)}$

$$E(X) = (1)(0.5) + (2)(0.5)$$

$$= 3/2$$

$$\frac{1}{E(X)} = \frac{2}{3}$$

$$\neq E(\tfrac{1}{X})$$

E.g. 3

Let x and y be jointly discrete with PMF

x \ y	0	1
0	0.2	0.1
1	0.4	0.3

$$E(xy) = \sum_{x,y} (x)(y)\, P_{x,y}(x,y)$$

$$= (0)(0)(0.2) + (0)(1)(0.1) + (1)(0)(0.4) + (1)(1)(0.3)$$

$$= 0.3$$

Note: $E(xy) = E(x) \cdot E(y)$ if x and y are independent.
$E(xy) \neq E(x) \cdot E(y)$ if x and y are not independent.

- Let x and y be discrete r.v. and let $h: R^2 \to R^1$ be some function s.t. the expectation of the r.v. $h(x,y)$ exists. Then,

$$E(h(x,y)) = \sum_{x,y} h(x,y)\, P_{x,y}(x,y)$$

- Linearity of Expected Value:
Let x and y be discrete r.v. and let a and b be real numbers. Then, $E(ax+by) = aE(x) + bE(y)$.

$$E(x+y) = E(x) + E(y)$$

$$E(ax) = aE(x)$$

- $$E\left[\sum_{i=1}^{k} x_i\right] = \sum_{i=1}^{k} E(x)$$

- Monotoncity of Expectation:

Let X and Y be discrete r.v. and suppose $X \le Y$.
Then, $E(X) \le E(Y)$.

2. The Abs Cont Case:

- If X is an abs cont r.v., then its expectation is

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

E.g. 1 $X \sim Exp(\lambda)$. Find $E(X)$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^{\infty} (x) \lambda e^{-\lambda x}\, dx$$

Let $u = x$. $du = dx$
Let $dv = \lambda e^{-\lambda x}$. $v = -e^{-\lambda x}$

$$= uv - \int_0^{\infty} v\, du$$

$$= \left[-xe^{-\lambda x} \Big|_0^{\infty} \right] + \int_0^{\infty} e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda}$$

- Let X be an abs cont r.v. with density function $f_X(x)$, and let $g: R \to R$ be some function. Then,

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \, dx$$

- Let x and y be jointly abs cont with joint density function $f_{x,y}(x,y)$, and let $h: R^2 \to R$ be some function. Then,

$$E(h(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) \, f_{x,y}(x,y) \, dx\,dy$$

- Linearity of Expected Values:

  Let x and y be jointly abs cont r.v.
  Let a and $b \in R$.
  $E(ax+by) = aE(x) + bE(y)$

  Note: $E(x+y) = E(x) + E(y)$
  $E(ax) = aE(x)$

- Let x and y be jointly abs cont r.v.
  If x and y are independent, $E(xy) = E(x) \cdot E(y)$
  If x and y are not indep, $E(xy) \neq E(x) \cdot E(y)$

- Monotonicity of Expected Values:

  Let X and Y be jointly abs cont r.v. with $X \leq Y$. Then, $E(X) \leq E(Y)$.

Expectations of Abs Cont Distributions:

1. $X \sim \text{Uniform}[L,R] \rightarrow E(X) = \frac{R+L}{2}$

2. $X \sim \text{Exp}(\lambda) \rightarrow E(X) = \frac{1}{\lambda}$

3. $X \sim N(0,1) \rightarrow E(X) = 0$

4. $X \sim N(\mu, \sigma^2) \rightarrow E(X) = \mu$

5. $X \sim \text{Beta}(a,b) \rightarrow E(X) = \frac{a}{a+b}$

6. $X \sim \text{Gamma}(\alpha, \lambda) \rightarrow E(X) = \frac{\alpha}{\lambda}$

E.g. 2. Let X and Y have joint probability function given

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{2}, & x=2,\ y=10 \\ \frac{1}{6}, & x=-7,\ y=10 \\ \frac{1}{12}, & x=2,\ y=12 \\ \frac{1}{12}, & x=-7,\ y=12 \\ \frac{1}{12}, & x=2,\ y=14 \\ \frac{1}{12}, & x=-7,\ y=14 \\ 0, & \text{otherwise} \end{cases}$$

Find $E(X^2)$

$$E(X^2) = \sum_x x^2 P_{X,Y}(x,y)$$

$$= (2)^2\left(\tfrac{1}{2}\right) + (-7)^2\left(\tfrac{1}{6}\right) + (2)^2\left(\tfrac{1}{12}\right) + (-7)^2\left(\tfrac{1}{12}\right) + (2)^2\left(\tfrac{1}{12}\right) + (-7)^2\left(\tfrac{1}{12}\right)$$

$$= 19$$