Conditional Distributions

Recall that $P(x|y) = \frac{P(xy)}{P(y)}$ and $P(y|x) = \frac{P(xy)}{P(x)}$.

1. Definition:

Let X and Y be jointly discrete r.v. Then, for any x s.t. $P(X=x)>0$, the cond dist of Y given $X=x$ is the probability dist assigning probability $\frac{P(Y\in B, X=x)}{P(X=x)}$ to each event $Y \in B$, $\forall B \subseteq \mathbb{R}$.

In particular, it assigns probability $\frac{P(a<Y\leq b, X=x)}{P(X=x)}$ to the event that $a<Y\leq b$.

2. Conditional Mass Function:

Let X and Y be jointly discrete with joint mass function $P_{X,Y}$ and let $x \in \mathbb{R}$ be s.t. $P(X=x)>0$. Then, the cond mass function of Y given $X=x$ is given by

$$P_{Y|X}^{y|x} = \frac{P_{X,Y}(x,y)}{P(X=x)} = \frac{P(Y=y, X=x)}{P(X=x)}$$

3. Conditional Density Function:

Let X and Y be jointly abs cont with joint density function $f_{X,Y}$ and let $x \in \mathbb{R}$ s.t. $f_X(x)>0$. Then, the cond density function of Y given $X=x$ is given by

$$f_{Y|X}^{y|x} = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

4. Conditional Distribution with Continuous R.V.

Let X and y be jointly abs rv with joint density function $f_{x,y}$. The conditional distribution of Y, given $X=x$, is defined by

$$P(a \le y \le b \mid X=x) = \int_a^b f_{y|x}^{(y|x)}\, dy, \quad \text{with } a \le b$$

and $f_x^{(x)} > 0$.

E.g. $f_{x,y}^{(x,y)} = \begin{cases} 4xy, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

a) Find the cond density function of y given $X=x$.

$$f_{y|x}^{y|x} = \frac{f_{x,y}^{(x,y)}}{f_x^{(x)}} = \frac{4xy}{2x} = 2y$$

$$f_x^{(x)} = \int_0^1 4xy\, dy = 4\int_0^1 xy\, dy = 4\left[x\left(\frac{y^2}{2}\Big|_0^1\right)\right] = 2x$$

$$f_{y|x}^{y|x} = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

b) Find $f_{x|y}^{x|y}$.

$$f_{x|y}^{x|y} = \frac{f_{x,y}^{(x,y)}}{f_y^{(y)}} = \frac{4xy}{2x} = 2y$$

$$f_{x|y}^{x|y} = \begin{cases} 2y, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$