Approximations

1. Relationship With Taylor Series:

- Taylor Series can be used to approximate functions.

- Recall the formula for linear approx:
$$L = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$
This is the same as T_1 for Taylor Series.

- T_1 is linear approx.
T_2 is quad approx.
The larger n is, the more closely T_n will approximate a function.

2. Examples:

1. Find the linear and quadratic approx for $\tan\left(\frac{\pi + 0.01}{3.97}\right)$.

Soln:
Let $f(x,y) = \tan\left(\frac{\pi}{4}\right)$ and $(x_0, y_0) = (\pi, 4)$.

Then, $\tan\left(\frac{\pi+0.01}{3.97}\right) = f(\pi+0.01, 4-0.03)$

$f(\pi,4) = 1$

$f_x(\pi,4) = \frac{1}{2}$

$f_y(\pi,4) = -\frac{\pi}{8}$

$f_{xx}(\pi,4) = \frac{1}{4}$

$f_{xy}(\pi,4) = -\frac{\pi^2}{16} - \frac{1}{8}$

$f_{yy}(\pi,4) = \frac{\pi^2+4\pi}{64}$

The linear approx is given by T_1.

$$T_1 = f(\pi,4) + f_x(\pi,4)(x-x_0) + f_y(\pi,4)(y-y_0)$$
$$= 1 + \frac{1}{2}(\pi+0.01-\pi) - \frac{\pi}{8}(4-0.03-4)$$
$$= 1 + \frac{1}{2}(0.01) - \frac{\pi}{8}(-0.03)$$
$$\approx 1.01678097245$$

The quad approx is given by T_2.

$$T_2 = f(\pi,4) + f_x(\pi,4)(x-x_0) + f_y(\pi,4)(y-y_0) + \left(\frac{1}{2!}\right) f_{xx}(\pi,4)(x-x_0)^2 + \left(\frac{1}{2!}\right)(2)(f_{xy}(\pi,4))(x-x_0)(y-y_0) + \frac{1}{2!} f_{yy}(\pi,4)(y-y_0)^2$$

$$= 1 + \frac{1}{2}(0.01) - \frac{\pi}{8}(-0.03) + \frac{1}{4}\left(\frac{1}{2}\right)(0.01)^2 + \left(-\frac{\pi^2}{16} - \frac{1}{8}\right)(0.01)(-0.03) + \left(\frac{1}{2}\right)\left(\frac{\pi^2+4\pi}{64}\right)(-0.03)^2$$

$$\approx 1.01704763026$$

Note:

$\tan\left(\frac{\pi+0.01}{3.97}\right) \approx 1.017052$

T_2 is much closer to the actual value than T_1.