Projections

1. Projection of b on sp(a):

The projection of b on sp(a), denoted by $proj_a b$, is equal to $\frac{a \cdot b}{\|a\|^2}[a]$.

E.g. 1. Find the projection p of the vector [1,2,3] on sp([2,4,3]) in R^3.

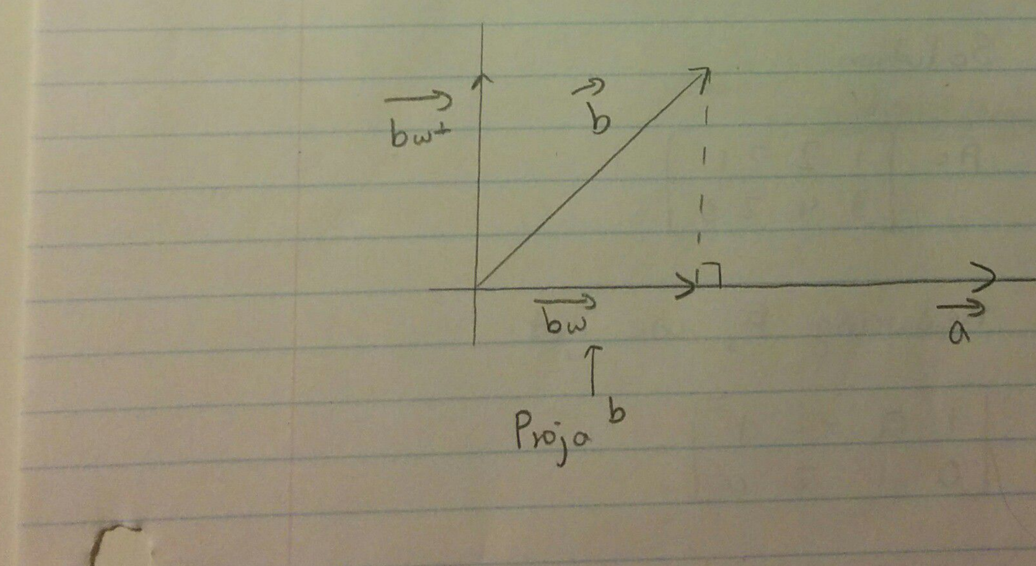
Solution:

$$p = \frac{a \cdot b}{\|a\|^2}[a]$$

$$= \frac{[2,4,3] \cdot [1,2,3]}{(2)^2 + (4)^2 + (3)^2}[2,4,3]$$

$$= \frac{2+8+9}{4+16+9}[2,4,3]$$

$$= \frac{19}{29}[2,4,3]$$



b can be broken down into 2 parts, bw and $bw^\perp$. bw is parallel to A while $bw^\perp$ is orthogonal to A.

$b = bw + bw^\perp$

2. Orthogonal Complement:

Let W be a subspace of R^n. The set of all vectors in R^n that are orthogonal to every vector in W is the orthogonal complement of W, and is denoted by $W^\perp$.

How to find $W^\perp$:

1. Find a matrix A having row vectors as a generating set for W.

2. Find the Nullspace of A.

E.g. 2. Find a basis for the ortho comp in R^4 of the subspace $W = sp([1,2,2,1], [3,4,2,3])$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 2 & 3 \end{bmatrix}$$

Reducing A, we get

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Let $x_3 = s$

Let $x_4 = t$

$x_2 + 2x_3 = 0$
$x_2 = -2x_3$
$= -2s$

$x_1 - 2x_3 + x_4 = 0$
$x_1 = 2x_3 - x_4$
$= 2s - t$

Answer: sp([2, -2, 1, 0], [-1, 0, 0, 1])

Properties of $W^\perp$:
Suppose $\dim(W) = k$, then

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $\dim(W^\perp) = n-k$, where n is from $\mathbb{R}^n$.
3. $(W^\perp)^\perp = W$
4. Each vector b in $\mathbb{R}^n$ can be uniquely decomposed into b_W and $b_{W^\perp}$.

3. Projection of a vector onto a subspace:

Let b be a vector in $\mathbb{R}^n$ and let W be a subspace in $\mathbb{R}^n$.

1. Find a basis for W.
2. Find a basis for $W^\perp$.
3. Find the coordinate vector r of b relative to the basis of W.
4. $b_W = r_1 v_1 + r_2 v_2 + \ldots$ where r_i is from r and v_i is from the basis of W.

E.g. 3 Find the projection of $b = [2,1,5]$ on the subspace of $W = sp([1,2,1], [2,1,-1])$.

Solution:

1. Since $[1,2,1]$ and $[2,1,-1]$ are linearly indep, they serve as the basis for W.

2. $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{matrix} \leftarrow V_1 \\ \leftarrow V_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Let $x_3 = s$

$-x_2 - x_3 = 0$

$-x_2 = x_3$

$x_2 = -x_3$

$= -s$

$x_1 + 2x_2 + x_3 = 0$

$x_1 = s$

$V_3 = [1,-1,1]$

3. $$\begin{array}{cccc} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & -1 & 1 & 5 \end{array}\right] \\ V_1 \quad V_2 \quad V_3 \quad b \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$b_W = 2V_1 + (-1)V_2$

$= 2[1,2,1] - [2,1,-1]$

$= [0,3,3]$

Note: $2v_3 = [2,-2,2]$ is the projection of b on $W^\perp$, which is $b_{W^\perp}$.

$b = b_W + b_{W^\perp}$
$= [0,3,3] + [2,-2,2]$
$= [2,1,5]$

4. Projection of a vector onto a plane.

E.g. 4 Find the projection of the vector $[3,-1,2]$ on the plane $x+y+z=0$ through the origin in $\mathbb{R}^3$.

Let $b = [3,-1,2]$
Let $a = [1,1,1]$ (Take the coefficient of x,y,z.)

$$b_{W^\perp} = \frac{a \cdot b}{\|a\|^2}[a]$$
$$= \frac{[3,-1,2]\cdot[1,1,1]}{1^2+1^2+1^2}[1,1,1]$$
$$= \frac{4}{3}[1,1,1]$$

$$b_W = b - b_{W^\perp}$$
$$= [3,-1,2] - \frac{4}{3}[1,1,1]$$
$$= \frac{1}{3}[5,-7,2]$$

Note: If you have a plane $ax+by+cz=d$, $[a,b,c]$ is perpendicular to that plane.

5. Projections in Inner Product Spaces

E.g. 5 Let the inner product of 2 polynominals $P(x)$ and $g(x)$ in the space $P_{0,1}$ of polynominal functions with domain $0 \le x \le 1$ be defined by $\langle f(x), g(x) \rangle = \int_0^1 f(x)\,g(x)\,dx$.

Find the projection of $f(x) = x$ on $sp(1)$.

Solution:

$$\frac{\int_0^1 (x)(1)\,dx}{\int_0^1 (1)(1)\,dx}$$

$$= \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx}$$

$$= \frac{\left[\frac{x^2}{2}\Big|_0^1\right]}{\left[x\Big|_0^1\right]}$$

$$= \frac{\frac{1}{2}}{1}$$

$$= \frac{1}{2}$$