Orthogonal Matrices

1. Definition and Properties:

Let A be a $n \times n$ matrix with col vectors $a_1, a_2, \ldots a_n$. These vectors form an orthonormal basis for R^n iff:

1. $a_i \cdot a_j = 0 \quad \forall i \neq j$
2. $a_i \cdot a_i = 1 \quad \forall i$

This implies that the cols of A form an orthonormal basis of R^n iff $A^T A = I$.

An $n \times n$ matrix is orthogonal if $A^T A = I$ and each col must also be unit vectors. I.e. Each col must have length 1.

Properties:
Let A be an $n \times n$ matrix. The following are equivalent.

1. The rows of A form an orthonormal basis for R^n.

2. The cols of A form an orthonormal basis for R^n.

3. A is invertible with $A^{-1} = A^T$.

E.g. 1 Verify that the matrix

$$A = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

is an orthogonal matrix and find A^{-1}.

Solution:

$$A^T = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$A^T A = \frac{1}{49}\begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Furthermore, each col must be of length 1.

Col 1. $\frac{1}{7}\left(\sqrt{2^2+3^2+6^2}\right) = \frac{1}{7}\left(\sqrt{49}\right) = \frac{7}{7} = 1$

Col 2. $\frac{1}{7}\left(\sqrt{3^2+(-6)^2+2^2}\right) = 1$

Col 3. $\frac{1}{7}\left(\sqrt{6^2+2^2+3^2}\right) = 1$

$\therefore$ A is an orthogonal matrix.

Since A is an orthogonal matrix,

$A^{-1} = A^T$.

2. Properties of Ax for an Orthogonal Matrix A:

Let A be an orthogonal $n \times n$ matrix and let x and y be any col vectors in R^n.

1. $(Ax) \cdot (Ay) = x \cdot y$
2. $\|Ax\| = \|x\|$
3. The angle btwn non-zero vectors x and y equal the angle btwn Ax and Ay.

E.g. 2 Let v be a vector in R^3 with coordinate vector $[2, 3, 5]$ relative to some ordered basis (a_1, a_2, a_3) of R^3. Find $\|v\|$.

Solution:

$V = 2a_1 + 3a_2 + 5a_3$

$V = Ax$, where $A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$

and $x = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.

$\|v\| = \|x\|$
$= \sqrt{2^2 + 3^2 + 5^2}$
$= \sqrt{38}$

3. Orthogonal Diagonizalization of Real Symmetric Matrices:

Eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal.

Every real symmetric matrix A is diagonalizable. The diagonalization $D = C^{-1}AC$ can be achieved by using a real orthogonal matrix C.

Note, if $D = C^{-1}AC$ is a diagonal matrix and C is an orthogonal matrix, then A is symmetric.

E.g. 3. Find an orthogonal diagonalization of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$$0 = \det(A - \lambda I)$$
$$= \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= \lambda^2 - 5\lambda$$
$$= \lambda(\lambda - 5)$$

$\lambda_1 = 0$ or $\lambda_2 = 5$

1. $\lambda_1 = 0$

$$A - \lambda_1 I$$
$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Let $x_2 = s$

$x_1 = -2s$

$E_1 = sp\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$

2. $\lambda_2 = 5$

$$A - \lambda_2 I$$

$$= \begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Let $x_2 = s$

$2x_1 = s$

$x_1 = \frac{s}{2}$

$$E_2 = sp\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

This is the diagonalization of A.

$$\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

4. Orthogonal Linear Transformations:

A linear transformation $T: R^n \rightarrow R^n$ is orthogonal if it satisifies $T(v) \cdot T(w) = v \cdot w$ for all v and w in R^n.

Properties:
1. $T(x) \cdot T(y) = x \cdot y$
2. $\|T(x)\| = \|x\|$
3. The angle btwn $T(x)$ and $T(y)$ equals the angle btwn x and y.

5. Orthogonal Transformations vis-à-vis Matrices:

A linear transformation T of R^n into itself is orthogonal iff its standard matrix representation A is an orthogonal matrix.

E.g. Show that the linear transformation $T: R^3 \rightarrow R^3$ defined by $T([x_1, x_2, x_3]) = [x_1/\sqrt{2} + x_3/\sqrt{2},\ x_2,\ -x_1/\sqrt{2} + x_3/\sqrt{2}]$ is orthogonal.

Solution:

The standard matrix rep is $\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

is an orthogonal matrix.
$\therefore$ The lin trans is orthogonal.