Gram - Schmidt Process

1. Orthogonal Bases:

A set $\{v_1, v_2, \ldots v_k\}$ of non-zero vectors in R^n is orthogonal if $v_i \cdot v_j = 0 \ \forall i \neq j$.

E.g. 1 Find an ortho basis for the plane $2x - y + z = 0$ in R^3.

Solution:

at $y = 0$ and $z = 2$. x works out to be -1.

$v_1 = [-1, 0, 2]$

Taking the coefficient of the plane, we get $[2, -1, 1]$.

$$v_2 = [-1, 0, 2] \times [2, -1, 1]$$
$$= [2, 5, 1]$$

The ortho basis of the plane is $\{v_1, v_2\}$.

Note: Since we are finding an ortho basis for a plane in R^3, we only need 2 vectors.

2. Projection Using an Ortho Basis:

Let $\{v_1, v_2, \dots v_k\}$ be an ortho basis for a subspace W in R^n. Let b be any vector in R^n.
The projection of b on W is:

$$b_W = \frac{b \cdot v_1}{||v_1||^2} v_1 + \frac{b \cdot v_2}{||v_2||^2} v_2 + \dots \frac{b \cdot v_k}{||v_k||^2} v_k$$

E.g. 2. Find the projection of $b = [3,-2,2]$ on the plane $2x-y+z=0$ in R^3.

From 1, we found the ortho basis of the plane to be $\{[-1,0,2], [2,5,1]\}$.

$$b_W = \frac{b \cdot v_1}{||v_1||^2} v_1 + \frac{b \cdot v_2}{||v_2||^2} v_2$$

$$= \frac{[3,-2,2] \cdot [-1,0,2]}{(-1)^2 + (2)^2} [-1,0,2] +$$

$$\frac{[3,-2,2] \cdot [2,5,1]}{(2)^2 + (5)^2 + (1)^2} [2,5,1]$$

$$= \frac{1}{5} [-1,0,2] - \frac{2}{30} [2,5,1]$$

$$= \left[-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$$

3. Orthonormal Basis:

Let W be a subspace of R^n. A basis $\{q_1, q_2, \dots q_k\}$ for W is orthonormal if:

1. $q_i \cdot q_j = 0 \ \forall i \neq j$
2. $q_i \cdot q_i = 1$

4. Projection of b on W with orthonormal basis:

$$b_W = (b \cdot q_1) q_1 + (b \cdot q_2) q_2 + \dots$$

Note: $q_1 = \frac{V_1}{||V_1||^2}$, $q_2 = \frac{V_2}{||V_2||^2}$, $\dots$ $q_k = \frac{V_k}{||V_k||^2}$

E.g. 3. Find an orthonormal basis for $W = sp(v_1, v_2, v_3)$ in R^4 if $v_1 = [1,1,1,1]$, $v_2 = [-1,1,-1,1]$, $v_3 = [1,-1,-1,1]$. Then, find the projection of $b = [1,2,3,4]$ on W

Solution

$V_1 \cdot V_2 = [1,1,1,1] \cdot [-1,1,-1,1] = 0$

$V_1 \cdot V_3 = [1,1,1,1] \cdot [1,-1,-1,1] = 0$

$V_2 \cdot V_3 = [-1,1,-1,1] \cdot [1,-1,-1,1] = 0$

Since $||V_1|| = ||V_2|| = ||V_3|| = 2$,

Let $q_1 = \frac{1}{2} V_1$, $q_2 = \frac{1}{2} V_2$, $q_3 = \frac{1}{2} V_3$.

Then, $q_1 \cdot q_1 = 1$, $q_2 \cdot q_2 = 1$, $q_3 \cdot q_3 = 1$.

$q_1 = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

$q_2 = [-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$

$q_3 = [\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$

The orthonormal basis for W is $\{q_1, q_2, q_3\}$.

To find the projection of $b = [1,2,3,4]$ on W, we do

$$
\begin{aligned}
b_W &= (b \cdot q_1)q_1 + (b \cdot q_2)q_2 + (b \cdot q_3)q_3 \\
&= \left([1,2,3,4] \cdot [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}]\right)\left([\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}]\right) + \left([1,2,3,4] \cdot [-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}]\right) \\
&\quad \left([-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}]\right) + \left([1,2,3,4] \cdot [\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}]\right)\left([\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}]\right) \\
&= [2,3,2,3]
\end{aligned}
$$

5. Gram-Schmidt Thm:

Let $\{a_1, a_2, \ldots a_k\}$ be a basis of W.

We can use this thm to find an orthogonal basis of W. Let $\{v_1, v_2, \ldots v_k\}$ be an orthogonal basis for W.

$$v_1 = a_1$$

$$v_2 = a_2 - \left(\frac{a_2 \cdot v_1}{\|v_1\|^2} v_1\right)$$

$$v_3 = a_3 - \left(\frac{a_3 \cdot v_1}{\|v_1\|^2} v_1 + \frac{a_3 \cdot v_2}{\|v_2\|^2} v_2\right)$$

In general,

$$v_i = a_i - \left(\frac{a_i v_1}{\|v_1\|^2} v_1 + \frac{a_i v_2}{\|v_2\|^2} v_2 + \ldots \frac{a_i v_{i-1}}{\|v_{i-1}\|^2} v_{i-1}\right)$$

To find the orthonormal basis, let $q_i = \left(\frac{v_i}{\|v_i\|}\right)$

E.g. 4. Find an orthogonal and orthonormal for $W = sp([1,0,1],[1,1,1])$ of $\mathbb{R}^3$.

$a_1 = [1,0,1] \rightarrow v_1 = a_1 = [1,0,1]$
$a_2 = [1,1,1]$

$$v_2 = a_2 - \left(\frac{a_2 \cdot v_1}{\|v_1\|^2} v_1\right)$$

$$= [1,1,1] - \frac{[1,1,1]\cdot[1,0,1]}{2}[1,0,1]$$

$$= [1,1,1] - [1,0,1]$$
$$= [0,1,0]$$

The orthogonal basis of W is $\{[1,0,1],[0,1,0]\}$.

To find the orthonormal basis:

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{[1,0,1]}{\sqrt{(1)^2+(1)^2+0^2}} = \frac{1}{\sqrt{2}}[1,0,1]$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{[0,1,0]}{\sqrt{0^2+1^2+0^2}} = [0,1,0]$$

An orthonormal basis is $\left\{\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], [0,1,0]\right\}$.

6. QR- Decomposition:

Let A be an nxk matrix with indep col vectors in R^n. There exists an nxk matrix Q with orthonormal col vectors and an upper-triangular invertible kxk matrix R, s.t. A = QR.

E.g. 5.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. ← These are the vectors taken from E.g. 4.

$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ ← Comes from the calculations done in E.g. 4.

QR = A

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Find R ⇒

1. $[\frac{1}{\sqrt{2}}, 0] \cdot [r_{11}, 0] = 1$

$\frac{r_{11}}{\sqrt{2}} = 1$

$r_{11} = \sqrt{2}$

2. $[\frac{1}{\sqrt{2}}, 0] \cdot [r_{12}, r_{22}] = 1$

$\frac{r_{12}}{\sqrt{2}} = 1$

$r_{12} = \sqrt{2}$

3. $[0, 1] \cdot [r_{12}, r_{22}] = 1$

$r_{22} = 1$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

↑

QR=A