Partial Fraction Integrals

Type 1.

The multiplicity of all the factors of the denominator are 1.

E.g. $\int \frac{x-5}{(x+2)(x-1)}\,dx$, $\int \frac{x+2}{(x^2-3)(x+6)}\,dx$, $\int \frac{x}{(x^3-2)(x-6)}\,dx$

The integrals would be in this form:

$$\int \frac{h(x)}{(f(x))^1 (g(x))^1 \dots (m(x))^1}\,dx$$

where $f(x)$, $g(x)$ and all the other factors of the denominator have multiplicity 1 and can't be equal to 0.

How to Solve it:

Take $\int \frac{x+5}{(x+2)(x-1)}\,dx$ as an example.

Step 1.

Split the fraction up such that each factor becomes a denominator and write each numerator in a general term such that the degree of the numerator is 1 less than the degree of its denominator.

$$\frac{A}{x+2} + \frac{B}{x-1}$$

Since $x+2$ has a degree of 1, then its numerator should have a degree of 0, or is a constant.

Since $x-1$ also has a degree of 1, its numerator will be a constant.

Note: Do not use a variable more than once. If you used A, then you can't use it again. Also, if the denominator isn't factored, you must factor it first.

$$\frac{A}{x-1}, \quad \frac{Ax+B}{x^2+3}, \quad \frac{Ax^2+Bx+C}{x^3+4}, \quad \frac{Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots Z}{x^n + \text{a constant}},$$ Z is a constant

If the denominator is linear, then its numerator is a constant.
If the denominator is quadratic, then its numerator is linear.
If the denominator is cubic, then its numerator is quadratic.
If the denominator is quartic, then its numerator is cubic.
And so on.

Step 2.
Add the fractions up. Then, expand the numerator and group like terms.

$$\frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$= \frac{Ax-A+Bx+2B}{(x+2)(x-1)}$$ ← Expanded the numerator

$$= \frac{(A+B)x-A+2B}{(x+2)(x-1)}$$ ← Since I have $Ax+Bx$, I can group them to be $(A+B)x$.

Step 3.
Compare your fraction with the original fraction. Then, create a system of equations and solve for A, B, ...

$$\frac{(A+B)x-A+2B}{(x+2)(x-1)} = \frac{x+5}{(x+2)(x-1)}$$ ← The denominators are the same, so we can ignore them.

$$(A+B)x - A + 2B = x+5$$ ← Here, since the coefficient of x is 1, $A+B=1$. Furthermore, because the constant is 5, $2B-A=5$.

The 2 equations, Solve for A and B now.

$A+B=1$

$2B-A=5$

$A=1-B$

$2B-(1-B)=5$

$2B-1+B=5$

$3B=6$

$B=2$

$A=-1$

Step 4.

Now that we have A and B, plug them into their fraction and integrate each fraction.

$$\int \frac{x+5}{(x+2)(x-1)}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-1}\,dx$$

$$= \int \frac{-1}{x+2}\,dx + \int \frac{2}{x-1}\,dx$$

$$= -\ln|x+2| + 2\ln|x-1| + C$$

Type 2.

One or more of the factors in the denominator has a multiplicity greater than 1.

E.g. $\int \frac{x-5}{(x-2)^2(x+1)}\,dx$, $\int \frac{x+6}{(x^2-3)^3(x^2-6)^2}\,dx$

$(x-2)$ has a multiplicity of 2

(x^2-3) has a multiplicity of 3.

(x^2-6) has multiplicity of 2.

How to Solve it:

Take $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$ as an example.

Step 1.

Since the denominator isn't factored, we must factor it first.

$$\int \frac{5x^3-3x^2+2x-1}{(x)^2(x^2+1)}\,dx$$

Now, we can split up each factor into its own fraction.

However, if there is a factor with a multiplicity greater than 1, we have to write each of its terms, starting from multiplicity of 1 to the multiplicity of the factor.

In our case, $(x)^2$ has a multiplicity of 2.
Therefore, our fraction will look like: $\frac{A}{(x)} + \frac{B}{(x)^2} + \frac{Cx+D}{x^2+1}$

Note: Do not confuse multiplicity with the degree of the denominator. If you have $(x)^2$, then x has a degree of 1 and a multiplicity of 2. Therefore, its numerator is a constant. If you have $(x^2+1)^4$, the degree of the function is 2, but its multiplicity is 4. The degree is the highest power INSIDE the brackets. The multiplicity is the exponent OUTSIDE the brackets.

Since I had a factor, x, that has a multiplicity greater than 1, I listed the terms starting from a multiplicity of 1 to the multiplicity of the factor, 2.

E.g. $$\frac{5x}{(x-3)^3(x+6)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{(x+6)}$$

Has a multiplicity of 3.

Starts from a multiplicity of 1, $(x-3)$, then goes to a multiplicity of 2, $(x-3)^2$, and finally ends at a multiplicity of 3, the multiplicity of the factor.

E.g. $$\frac{5x}{(2x+1)^5(x-3)} = \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3} + \frac{D}{(2x+1)^4} + \frac{E}{(2x+1)^5} + \frac{F}{(x-3)}$$

In this case, I wrote the factor starting from multiplicity of 1 and ended at multiplicity of 5.

Note: When you are listing the factors from multiplicity of 1 to its end, each of those terms must have the same type of numerator.
I.e. If the numerator of the 1st term is a constant, then the rest of those terms will also have a numerator with a constant.

Step 2.
Add the fractions, expand the numerator and group like terms.

$$\frac{A}{(x)} + \frac{B}{(x)^2} + \frac{Cx+D}{x^2+1} = \frac{A(x)(x^2+1) + B(x^2+1) + ((Cx+D)(x^2)}{(x^2)(x^2+1)}$$

$$= \frac{A(x^3+x) + Bx^2 + B + Cx^3 + Dx^2}{(x^2)(x^2+1)}$$

$$= \frac{(A+C)x^3 + (B+D)x^2 + Ax + B}{(x^2)(x^2+1)}$$

Step 3.
Compare the new fraction with the original fraction, and solve for A, B, C, D.

$(A+C)x^3 + (B+D)x^2 + Ax + B = 5x^3 - 3x^2 + 2x - 1$

$A+C=5 \rightarrow C=3$

$B+D=-3 \rightarrow D=-2$

$A=2$

$B=-1$

Step 4.
Plug the values for A, B, C, D back into their fraction and integrate.

$$\int \frac{2}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{3x-2}{x^2+1}\,dx$$

$$= 2\ln|x| - x^{-1} + \int \frac{3x}{x^2+1}\,dx + \int \frac{-2}{x^2+1}\,dx$$

$$= 2\ln|x| - x^{-1} + \frac{3}{2}\ln|x^2+1| - 2\arctan(x) + C$$

Type 3.

So far, all our examples have been $f(x) = \frac{P(x)}{Q(x)}$, s.t. the degree of $P(x) <$ the degree of $Q(x)$. These are proper rational functions. However, if the degree of $P(x) \geq$ the degree of $Q(x)$, we have to do long division first.

Step 1.

Do long division.

Take $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$ as an example.

$$
\begin{array}{r l}
 & x + 1 \\
x^3 - x^2 - x + 1 \,\big) & x^4 + 0x^3 - 2x^2 + 4x + 1 \\
 & -(x^4 - x^3 - x^2 + x) \\
\hline
 & x^3 - x^2 + 3x + 1 \\
 & -(x^3 - x^2 - x + 1) \\
\hline
 & 4x
\end{array}
$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \frac{(x^3 - x^2 - x + 1)(x+1) + 4x}{x^3 - x^2 - x + 1}\,dx$$

$$= \int (x+1) + \frac{4x}{x^3 - x^2 - x + 1}\,dx$$

Now, solve the integral using type 1 or type 2 or other methods.

Note: If there is a term missing from the dividend, the polynominal in the $\int$, the we must add a 0(term) to it. E.g. $x^4 - x^2 + 4x + 1$ is missing an x^3 term, so I added $0x^3$ to it.

Furthermore, Dividend = (Divisor)(Quotient) + Remainder.

E.g. $\underbrace{x^4 - 2x^2 + 4x + 1}_{\text{Dividend}} = \underbrace{(x^3 - x^2 - x + 1)}_{\text{Divisor}}\underbrace{(x+1)}_{\text{Quotient}} + \underbrace{4x}_{\text{Remainder}}$

Trig Integrals

Useful Trig Indentities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $\cot^2 x + 1 = \csc^2 x$
4. $\cos^2 x = \dfrac{1+\cos(2x)}{2}$
5. $\sin^2 x = \dfrac{1-\cos(2x)}{2}$
6. $\sin(2x) = 2\sin x \cos x$

Type 1
Odd/Even Case

Type 2
odd/odd Case

Type 3
Even/ Even Case

Trig Substitution

Use it when you see any of these forms:
Let $a \in \mathbb{R}$, $u =$ a polynominal expression with a variable

1. $a^2 + u^2$ Constant squared + variable expression squared
   Let $u = a\tan\theta$

2. $a^2 - u^2$ Constant squared - variable expression squared
   Let $u = a\sin\theta$

3. $u^2 - a^2$ Variable expression squared - Constant squared
   Let $u = a\sec\theta$

E.g.

1. $\int \frac{3x+1}{\sqrt{x^2+9}}\,dx$

Because x^2+9 is of the form a^2+u^2, with $a=3$ and $u=x$, let $x = 3\tan\theta$.

2. $\int \frac{(x+3)(4x^2-16)^{5/3}}{\sqrt{x}}\,dx$

Because $4x^2-16 = (2x)^2 - 4^2$, this is of the form u^2-a^2, with $a=4$ and $u=2x$, let $2x = 4\sec\theta$

3. $\int \frac{1}{\sqrt{2-(3x+1)^2}}\,dx$

$2-(3x+1)^2 = (\sqrt{2})^2 - (3x+1)^2$. This is of the form a^2-u^2, with $a=\sqrt{2}$ and $u=3x+1$. Let $3x+1 = \sqrt{2}\sin\theta$

Note: Trig sub is the longest way to solve an integer. If there's a faster way, do that.

E.g.

$\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

This is of the form a^2+u^2, with $a=1$ and $u=x$.

Let $x = a\tan\theta$

$= \tan\theta$

$dx = \sec^2\theta\,d\theta$

Note: $x=\tan\theta$, so $x^2 = \tan^2\theta$ and $x^2+1 = \tan^2\theta+1 = \sec^2\theta$

$$\int \frac{\tan^2\theta}{\sqrt{\sec^2\theta}} \sec^2\theta \, d\theta$$

$$= \int \frac{\tan^2\theta}{|\sec\theta|} \sec^2\theta \, d\theta$$

$\sec\theta > 0$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

If you write this ↑, you can assume |trig expression| = postive trig exp.

$$= \int \frac{\tan^2\theta}{\sec\theta} \sec^2\theta \, d\theta$$

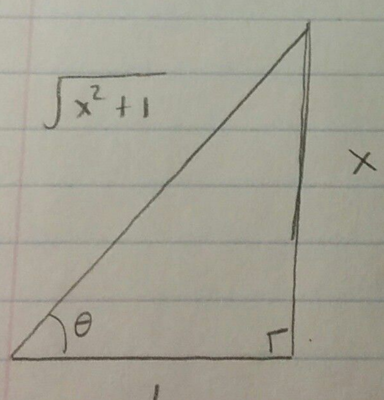
$$= \int \tan^2\theta \cdot \sec\theta \, d\theta$$

$$= \int (\sec^2\theta - 1)(\sec\theta)\, d\theta$$

$$= \int \sec^3\theta - \sec\theta \, d\theta$$

$$= \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta$$

$$= \frac{\sec\theta\tan\theta - \ln|\tan\theta + \sec\theta|}{2} + C$$



We said that $\tan\theta = x$

at the start.

$$\sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{1}{\frac{1}{\sqrt{x^2+1}}}$$

$$= \sqrt{x^2+1}$$

$$\frac{\sec\theta\tan\theta - \ln|\tan\theta + \sec\theta|}{2} + C$$

$$= \frac{(\sqrt{x^2+1})(x) - \ln|x + \sqrt{x^2+1}|}{2} + C$$

Integral List

1. $\int a\,dx = ax + c$, a is a constant

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$

3. $\int \frac{1}{x}\,dx = \ln|x| + c$

4. $\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$

5. $\int a^x\,dx = \frac{a^x}{\ln(a)} + c$

6. $\int \cos x\,dx = \sin x + c$

7. $\int \sin x\,dx = -\cos x + c$

8. $\int \sec^2 x\,dx = \tan x + c$

9. $\int \sec x \cdot \tan x\,dx = \sec x + c$

10. $\int \csc^2 x\,dx = -\cot x + c$

11. $\int \csc x \cdot \cot x\,dx = -\csc x + c$

12. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

13. $\int \csc x\,dx = -\ln|\csc x + \cot x| + c$

14. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$

15. $\int \frac{1}{1+x^2}\,dx = \arctan(x) + c$

16. $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + c$

17. $\int \ln(x)\,dx = x - x\ln(x) + c$