Calculus Notes

1. Summation

$$\sum_{i=m}^{n} a_k$$

i is the index
m is the inital value
n is the ending value
a_k is the function / general term

* Note *
Even though m can be less than 0, for our course, m must be greater than or equal to 0.

Properties of summation.

1.
$$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Proof:

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1+b_1) + (a_2+b_2) + \ldots + (a_n+b_n) \quad \text{By def of } \Sigma$$

$$= (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 \ldots + b_n)$$

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

QED

2.
$$\sum_{k=1}^{n} c(a_k) = c\sum_{k=1}^{n} a_k$$

Proof:

$$\sum_{k=1}^{n} c(a_k) = ca_1 + ca_2 \ldots + ca_n \quad (\text{By def of } \Sigma)$$

$$= c(a_1 + a_2 + \ldots + a_n)$$

$$= c\sum_{k=1}^{n} a_k$$

QED

3. $\sum_{k=1}^{n} a_k = \sum_{k=1}^{\ell-1} a_k + \sum_{k=\ell}^{n} a_k$

Proof:

$LS = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots a_n$

$RS = \sum_{k=1}^{\ell-1} a_k = a_1 + a_2 + \dots a_{\ell-1}$

$\sum_{k=\ell}^{n} a_k = a_\ell + a_{\ell+1} \dots + a_n$

$\sum_{k=1}^{\ell-1} a_k + \sum_{k=\ell}^{n} a_k = a_1 + a_2 \dots a_n$
$= LS$

QED

Sum Formulas

1. $\sum_{k=1}^{n} c = cn$, c is a constant

Proof:

$\sum_{k=1}^{n} c = c + c + c \dots$ (n times)

$= cn$

QED

2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

You can prove these with induction.

2. Riemann's Sum

A Riemann's Sum for a function f on the interval [a,b] is a sum of the form

$$\sum_{k=1}^{n} f(x_i^*) \Delta x$$

$\Delta x = \frac{b-a}{n}$ $x_i = a + i\Delta x$ $x_i^* \in [x_{i-1}, x_i]$

Left Riemann Sum

$$\sum_{k=1}^{n} f(x_{k-1}) \Delta x$$

Mid-Point Riemann Sum

$$\sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right) \Delta x$$

Right Riemann Sum

$$\sum_{k=1}^{n} f(x_k) \Delta x$$

Upper and Lower Sums

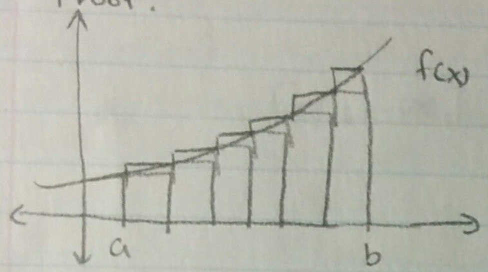
$$u(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

M_i is the sup of each partition.
m_i is the inf of each partition.

Using left Riemann's Sum and right Riemann's Sum and the def of increasing to prove that if f is positive and increasing on $[a,b]$ prove that left sum $\leq$ right sum.

Proof:



Let $P = \{x_i\}_{i=1}^{n}$ be any partition of $[a,b]$.

Left Riemann Sum $= \sum_{k=1}^{n} f(x_{i-1})\Delta x$

Right Riemann Sum $= \sum_{k=1}^{n} (f(x_i))\Delta x$

Since $x_{i-1} \leq x_i$, $f(x_{i-1}) \leq f(x_i)$, $\Delta x\, f(x_{i-1}) \leq \Delta x\, f(x_i)$,

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x \leq \sum_{i=1}^{n} f(x_i)\Delta x$$

$\therefore$ Left Riemann's Sum $\leq$ Right Riemann's Sum

Definite Integrals

Let f be a function defined on $[a,b]$. The definite integral of f from $x=a$ to $x=b$ is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

Properties of Definite Integral

Let $a,b \in \mathbb{R}$ s.t. $a<b$

Suppose f and g are integrable on $[a,b]$, then

1. If $f(x) \geq 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$
2. If $f(x) \leq 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \leq 0$
3. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$, k is a constant
5. $\int_a^a f(x)\,dx = 0$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $\forall c \in [a,b]$
7. If $\exists M, N \in \mathbb{R}^+$ s.t. $m \leq f(x) \leq n$, $\forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq n(b-a)$
8. If $f(x) \leq g(x)$ $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Definite Integral Formulas

1. $\int_a^b c\,dx = c(b-a)$, $c \in \mathbb{R}$ and c is a constant
2. $\int_a^b x\,dx = \frac{1}{2}(b^2-a^2)$
3. $\int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3)$

Darboux Definition

If $f(x)$ is cont on $[a,b]$ or if $f(x)$ has a finite number of jump discontunities, then f is integrable on $[a,b]$.

Consider $f(x) \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$ (Dirchlet Function)

Darboux's Def states that $f(x)$ is integrable on $[a,b]$ iff $\sup\{L(f,P) \mid \forall P \text{ of } [a,b]\} = \inf\{U(f,P) \mid \forall P \text{ of } [a,b]\}$

Integrability Reformation/ Darboux Integration states that f is integrable on $[a,b]$ iff $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$. s.t. $U(f,P) - L(f,P) < \varepsilon$.

FTOC Part 1

Let $a, b \in \mathbb{R}$ and $a<b$

If f is cont on $[a,b]$ and F is any antiderivate/integral of f on $[a,b]$, then $\int_a^b f(x)\,dx = f(b) - f(a)$.

Proof

Suppose f is cont on $[a,b]$ and F is an integral of f on $[a,b]$.

Let $P = \{x_i\}_{i=0}^{n}$ be any partition of $[a,b]$.

Let $x_i = a + i\Delta x$

Let $\Delta x = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \quad x_i^* \in [x_{i-1}, x_i]$$

F is diff on $[a,b]$ and cont on $[a,b]$ (diff $\Rightarrow$ cont).

F is diff and cont on each $[x_{i-1}, x_i] \in [a,b]$.

$\therefore$ MVT is applied to F on each partition.

By MVT, $\exists c \in [x_{i-1}, x_i]$ s.t. $F(c) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$

$$F(c)\underbrace{(x_i - x_{i-1})}_{\Delta x} = F(x_i) - F(x_{i-1})$$

$$\sum_{i=1}^{n} f(c)\Delta x = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$$

Let $x_i = c$

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(x_i)\Delta x$$

$= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) \ldots + (F(x_n) - F(x_{n-1}))$ (Telescoping sum)

$$= \underset{b}{\underset{\uparrow}{F(x_n)}} - \underset{a}{\underset{\uparrow}{F(x_0)}}$$

$= F(b) - F(a)$

QED

MVT for Integrals

Let $a, b \in \mathbb{R}$ s.t. $a < b$

If $f(x)$ is cont on $[a,b]$, then $\int_a^b f(x)\,dx = f(c)(b-a)$ for some $c \in [a,b]$.

Proof

Since f is cont on $[a,b]$, then if $\exists m, n \in \mathbb{R}^+$ s.t. $m \leq f(x) \leq n$, $\forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq n(b-a)$.

$$m(b-a) \leq \int_a^b f(x)\,dx \leq n(b-a)$$

$$m \leq \underbrace{\frac{\int_a^b f(x)\,dx}{b-a}}_{\text{Let this be } F} \leq n$$

By the Intermediate Value Theorem, we know $\exists c$ s.t. $m \leq c \leq n$.

$\therefore f(c) = F$

$$= \frac{\int_a^b f(x)\,dx}{b-a}$$

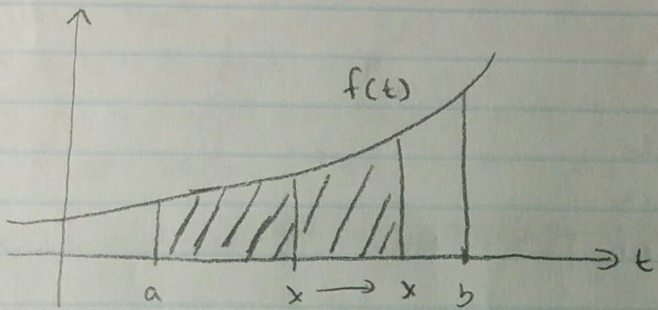
$$f(c)(b-a) = \int_a^b f(x)\,dx \longleftrightarrow \underbrace{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{Used to prove FTOC 2}}$$

QED

Area Accumlation Function

Suppose f is cont on $[a,b]$. Then, the area accumlation function for f on $[a,b]$ is the function that, for $x \in [a,b]$, is equal to the signed area between the graph of f and the x-axis on $[a,x]$:

$$A(x) = \int_a^x f(t)\,dt$$



As x increases, the integral of f(t) increases.
As x decreases, the integral of f(t) decreases.

FTOC 2

If f is cont on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt \;\forall x \in [a,b]$, then

a) F is cont and diff on $[a,b]$.

b) $F'(x) = f(x)$

Proof:

Suppose f is cont on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$

Case 1: $x \in (a,b)$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

If $h>0$,

$$\lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

If $h<0$,

$$\lim_{h\to 0} \frac{-\int_{x+h}^{x} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

If $h>0$

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

If $h<0$

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= -\int_{x+h}^a f(t)\,dt - \int_a^x f(t)\,dt$$

$$= -\int_{x+h}^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

$$\therefore \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right) \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} \left(\frac{1}{(x+h)-x}\right) \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c) \quad \text{(By MVT for Integrals)} \quad x \le c \le x+h$$

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c)$$

$$= f(x)$$

Case 2: $x=a$ and $x=b$

An analogus argument shows that the results are true; replace the 2-sided limit with 1-sided limit.

QED

Techniques of Integration

1. Inspection

If the integral is a basic integral, we can solve it by inspection.

E.g. $\int \cos x \, dx = \sin x + c$

2. Substitution

If f and g' are cont on $[a,b]$, then $\int_a^b f(x)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Proof:

Suppose f and g' are cont on $[a,b]$.
Let F be an integral of f on $[a,b]$.

$LS = \int_a^b f(x)g'(x)\,dx$
$= F(g(x)) \Big|_a^b$
$= F(g(b)) - F(g(a))$

$RS = \int_{g(a)}^{g(b)} f(u)\,du$
$= F(u) \Big|_{g(a)}^{g(b)}$
$= F(g(b)) - F(g(a))$

$LS = RS$

QED

3. Parts

If $u = f(x)$ and $v = g(x)$ are diff, then
$\int u \, dx = uv - \int v \, du$
$\int_a^b u \, dx = uv \Big|_a^b - \int_a^b v \, du$

Proof

Suppose $u = f(x)$ and $v = g(x)$ are diff
By product rule, $f'(x)g(x) + f(x)g'(x) = (fg)'$
$u'v + uv' = (uv)'$
$uv' = (uv)' - u'v$
$= (uv)' - vu'$
$\int uv' = \int (uv)' - \int vu'$

$\int u\,dv = \int (uv)' - \int v\,du$

$= uv - \int v\,du$

QED

Hints:

1. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$
3. Since a definite integral is almost always a real number, the derivative of a definite integral $= 0$, if $f(x)$ is cont.
4. For FTOC 1, we can assume that the function is cont. But, for FTOC 2 and other cases, we have to prove that the function is cont on $[a, b]$.

Improper Integrals

Type 1

When either the lower bound or the upper bound or both is infinity or negative infinity.

I.e.

$$\int_a^{\infty} f(x)\,dx, \quad \int_{-\infty}^{a} f(x)\,dx, \quad \int_{-\infty}^{\infty} f(x)\,dx$$

Case 1

When only 1 of the bounds is infinity or negative infinity

$$\int_a^{\infty} f(x)\,dx, \quad \int_{-\infty}^{a} f(x)\,dx$$

Rewrite the integral like this:

$$\lim_{A \to \infty} \int_a^{A} f(x)\,dx, \quad \lim_{A \to -\infty} \int_A^{a} f(x)\,dx$$

Then, solve the integral.

Finally, calculate the limit of your answer. If your final answer is a real number, then the integral converges and has a solution. If the final answer is positive or negative infinity, then the limit diverges.

Case 2

If both the upper and lower bounds are infinity or negative infinity.

$$\int_{-\infty}^{\infty} f(x)\,dx$$

Choose a number between ∞ and $-\infty$, preferably 0 if you can, and split the integral into 2 integrals.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{c}^{\infty} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx \qquad c \in (-\infty, \infty)$$

Then, evaluate each new integral like in Case 1. Finally, if either small integral diverges, then the entire thing diverges. If both small integrals converges, then the entire thing converges to their sum.

Type 2

When the integral has a V.A. within the interval of the lower bound to the upper bound, inclusive.

$$\int_0^3 \frac{1}{x}\,dx, \quad \int_0^4 \frac{1}{x-3}\,dx, \quad \int_0^4 \frac{1}{x-4}\,dx$$

$\frac{1}{x}$ has a V.A. at $x=0$, which is the lower bound.

$\frac{1}{x-3}$ has a V.A. at $x=3$, which is in the interval of $[0,4]$.

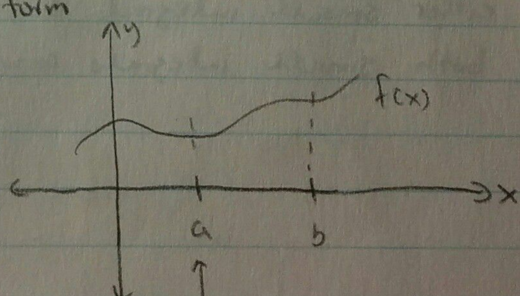
$\frac{1}{x-4}$ has a V.A. at $x=4$, which is the upper bound.

Case 1.

The V.A. is at the lower bound

Rewrite the integral in this form

$$\lim_{A \to a^+} \int_A^b f(x)\,dx$$



Solve the integral and then find the limit of your answer. If the final answer is a real number, the integral converges. If your final answer is $\pm\infty$, the integral div.

We find $\lim_{A \to a^+}$ because the integral is to the right of a.

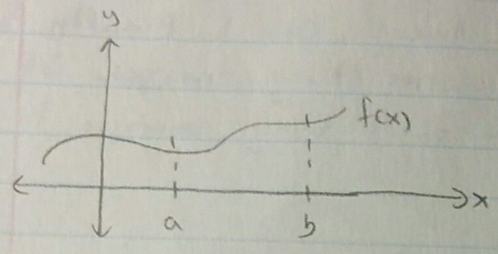
Case 2

The V.A. is at the upper bound.

This is similar to case1, but you do

$$\lim_{A \to b^-} \int_a^A f(x)\,dx$$ instead.

The rest of the steps are the same as case 1.



We find $\lim_{A \to b^-}$ because the integral is on the left of b.

Case 3

The V.A. is in (a,b).

Suppose the V.A. is c, and $c \in (a,b)$.

Split the integral up as such:

$$\int_a^b f(x)\,dx = \int_c^b f(x)dx + \int_a^c f(x)dx$$

Solve $\int_c^b f(x)\,dx$ using case 1.

Solve $\int_a^c f(x)\,dx$ using case 2.

If either small integral div, the whole thing div.

If both small integrals conv, the whole thing conv.

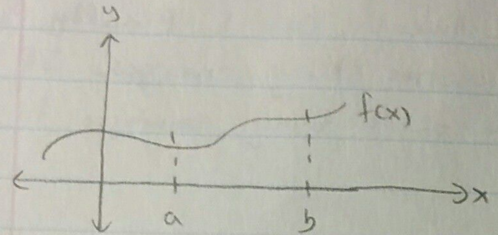
Case 2

The V.A. is at the upper bound.

This is similar to case1, but you do

$$\lim_{A \to b^-} \int_a^A f(x)\,dx$$ instead.

The rest of the steps are the same as case 1.



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$$\int_a^b f(x)\,dx = \int_c^b f(x)\,dx + \int_a^c f(x)\,dx$$

Solve $\int_c^b f(x)\,dx$ using case 1.

Solve $\int_a^c f(x)\,dx$ using case 2.

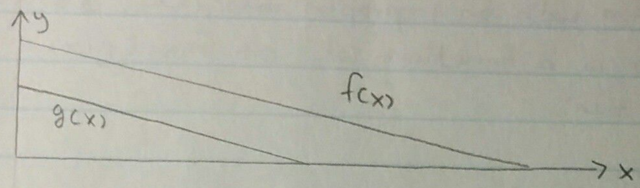
If either small integral div, the whole thing div.

If both small integrals conv, the whole thing conv.

Comparison Theorem

Suppose that $f(x)$ and $g(x)$ are functions that are cont on an interval, I.

1. If the improper integral of f on I converges and $0 \leq g(x) \leq f(x)$ $\forall x \in I$, then $g(x)$ also converges on I.



Since the area underneath $f(x)$ is finite, the area of $g(x)$ must also be finite.

Proof:

Suppose $f(x)$ is a function with an improper integral, $\int_a^\infty f(x)\,dx$, and it converges. If $g(x)$ is a function s.t. $0 \leq g(x) \leq f(x)$ $\forall x \in [a, \infty)$ and $B > a$, then:

$$0 \leq \int_a^B g(x)\,dx \leq \int_a^B f(x)\,dx$$

Let's take the limit as $B \to \infty$

$$0 \leq \lim_{B \to \infty} \int_a^B g(x)\,dx \leq \lim_{B \to \infty} \int_a^B f(x)\,dx$$

By def

$$0 \leq \int_a^\infty g(x)\,dx \leq \int_a^\infty f(x)\,dx$$

Since $\int_a^\infty f(x)\,dx$ conv to some number L, then

$\int_a^\infty g(x)\,dx$ must conv to a real number between 0 and L.

QED

2. If the improper integral of $f(x)$ on I div and $0 \leq f(x) \leq g(x)$ $\forall x \in I$, then the improper integral of g on I also div.

Proof:

Suppose $f(x)$ is a function with an improper integral, $\int_a^\infty f(x)\,dx$, and it diverges. If $g(x)$ is a function s.t. $0 \leq f(x) \leq g(x)$ $\forall x \in [a, \infty)$ and $B > a$, then:

$$0 \leq \int_a^B f(x)\,dx \leq \int_a^B g(x)\,dx$$

Let's take the limit as $B \to \infty$

$$0 \leq \lim_{B\to\infty} \int_a^B f(x)\,dx \leq \lim_{B\to\infty} \int_a^B g(x)\,dx$$

By def

$$0 \leq \int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

Since $\int_a^\infty f(x)\,dx$ div, then because $g(x) \geq f(x)$ $\forall x \in I$, $\int_a^\infty g(x)\,dx$ must div, too.

QED

Improper integrals of Power Functions on $[1, \infty)$

1. If $0 < p \leq 1$, then $\frac{1}{x^p} \geq \frac{1}{x}$ $\forall x \in [1, \infty)$ and $\int_1^\infty \frac{1}{x^p}\,dx$ div

2. If $p > 1$, then $\frac{1}{x^p} < \frac{1}{x}$ $\forall x \in [1, \infty)$ and $\int_1^\infty \frac{1}{x^p}\,dx$ conv to $\frac{1}{p-1}$.

Proof of 1

$\int_1^\infty \frac{1}{x}\,dx$

$= \lim_{A\to\infty} \int_1^A \frac{1}{x}\,dx$

$= \lim_{A\to\infty} \ln|x| \Big|_1^A$

$= \lim_{A\to\infty} \left(\ln|A| - \ln|1|\right)$

$= \lim_{A\to\infty} \ln|A|$

$= \infty$

$\therefore \int_1^\infty \frac{1}{x}\,dx$ div

By C.T., If $f(x)$ div and $0 \le f(x) \le g(x)$, then $g(x)$ also div.
If $0<p<1$, then $\frac{1}{x^p} > \frac{1}{x}$. $\therefore \int_1^\infty \frac{1}{x^p}\,dx$ div if $0<p\le 1$.

QED

Proof of 2

If $p>1$, then

$\int_1^\infty \frac{1}{x^p}\,dx$

$= \lim_{B\to\infty} \int_1^B x^{-p}\,dx$

$= \lim_{B\to\infty} \left[\frac{1}{1-p} x^{-p+1}\right]_1^B$

$= \lim_{B\to\infty} \left[\frac{1}{1-p} B^{1-p} - \frac{1}{1-p}\right]$

$= \frac{1}{p-1}$ QED

Improper Integrals of Power Functions on $[0,1]$

1. If $0<p<1$, then $\frac{1}{x^p} < \frac{1}{x}$ $\forall x \in [0,1]$ and $\int_0^1 \frac{1}{x^p}\,dx$ conv to $\frac{1}{1-p}$.

2. If $p \geq 1$, then $\frac{1}{x^p} \geq \frac{1}{x}$ $\forall x \in [0,1]$ and $\int_0^1 \frac{1}{x^p}\,dx$ div.

Proof of 1

If $0<p<1$, then

$$\int_0^1 \frac{1}{x^p}\,dx$$

$$= \lim_{A \to 0^+} \int_A^1 \frac{1}{x^p}\,dx$$

$$= \lim_{A \to 0^+} \left[\frac{x^{-p+1}}{1-p} \right]_A^1$$

$$= \lim_{A \to 0^+} \left[\frac{1}{1-p} \left(\underbrace{1^{-p+1}}_{\downarrow\, 1} - \underbrace{A^{-p+1}}_{\downarrow\, 0} \right) \right]$$

$$= \frac{1}{1-p}$$

QED

Proof of 2

$$\int_0^1 \frac{1}{x}\,dx$$

$$= \lim_{A \to 0^+} \int_A^1 \frac{1}{x}\,dx$$

$$= \lim_{A \to 0^+} \ln|x| \Big|_A^1$$

$$= \lim_{A \to 0^+} \ln|1| - \ln|A|$$

$$= \lim_{A \to 0^+} -\ln|A|$$

$$= \infty$$

$\therefore \int_0^1 \frac{1}{x}\,dx$ div

By C.T., if $f(x)$ div and $0 \leq f(x) \leq g(x)$ then $g(x)$ also div.

If $p>1$, then $\frac{1}{x^p} > \frac{1}{x}$.

$\therefore$ If $p \geq 1$, $\frac{1}{x^p}$ div.