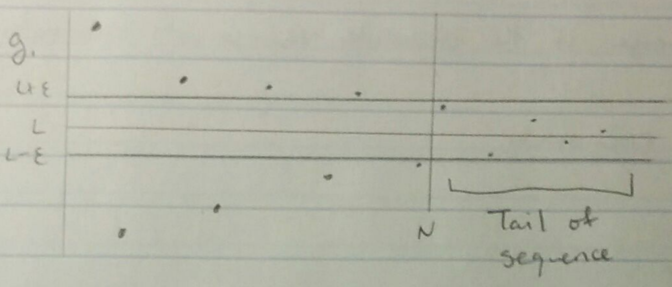
Sequence

- Tail of a sequence
  It is the part of the sequence that is after N.

E.g.



$\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L| < \varepsilon$

Based on this, if the tail of a sequence converges, then the entire sequence converges.

Series

- Denoted as $\sum_{n=1}^{\infty} a_n$, where a_n is the general term.

There are 8 ways to find if a series converges or diverges, but ONLY 2 will give you the sum.

1. By definition
   This will give you the sum of a converging series.
   Only use this if you have a telescoping sum.
   Add the telescoping sum and you will be left with only the first and last term. Then, take the limit of this term as $n \to \infty$. If the series converges, then your limit will be a real number. Otherwise, the series diverges.

2. Geometric Series Test

Will also give you the sum of a converging series.

Suppose $a_n = ar^n$

$\sum_{k=0}^{\infty} ar^n$ will converge if the absolute value of the ratio of

is between 0 and 1, exclusively.

I.e.

let r = ratio of r^n

If $0<|r|<1$, then the series converges and the sum $= \frac{a}{1-r}$.

Note that a is the first term of the series.

If $|r|>1$, then the series diverges.

Proof

let $a, r \in \mathbb{R} - \{0\}$

$a + ar + ar^2 + \ldots + ar^n = \sum_{n=0}^{\infty} ar^n$

r is the ratio

let $S_n = a + ar + ar^2 + \ldots ar^n$

let $rS_n = ar + ar^2 + ar^3 + \ldots ar^{n+1}$

$S_n - rS_n = a - ar^{n+1}$ (Telescoping sum)

$S_n(1-r) = a(1-r^{n+1})$

Case 1: $r \neq 1$

$$S_n = \frac{a}{1-r}\left(1-r^{n+1}\right)$$

$$= \frac{a}{1-r}\left(\lim_{n\to\infty}(1-r^{n+1})\right)$$

$$= \frac{a}{1-r}\left(1 - \lim_{n\to\infty} r^{n+1}\right)$$

$$\begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r|>1 \end{cases}$$

Case 2: $r=1$

$S_n = a + a + \ldots + a$

$= (n+1)(a)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (n+1)(a)$

$= \text{DNE}$

$\therefore$ Putting it together,

$$\begin{cases} \frac{a}{1-r}, & \text{if } |r|<1 \\ \text{DNE}, & \text{if } |r|\geq 1 \end{cases}$$

3. Divergence Test
Given a series, Σa_n, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.
Note: If $\lim_{n\to\infty} a_n = 0$, this DOES NOT mean the sequence convergences.

Proof: Contrapostive of Vanishing Condition

4. Integral Test
Given a series, Σa_n:
If I have a function, $f(x)$ such that 1. $f(n) = a_n$, $\forall n \in \mathbb{N}$
2. $f(n)$ is positive, $\forall n \in \mathbb{N}$
3. $f(x)$ is continuous, $\forall n \in \mathbb{N}$
4. $f(x)$ is decreasing on the interval

then, we can use integration to find if our series convergences or not.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \iff \int_1^{\infty} f(x)\,dx \text{ diverges}$$

Note: we don't have to worry about $f(x)$ being continuous, but we do have to prove the other 3.

To prove that $f(x)$ is positive, do a positivity check.

To prove that $f(x)$ is decreasing, find the derivative of $f(x)$ and prove that it's negative.

5. P- Series Test

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}^+$ is called a P- Series.

If $0 < p \leq 1$, then the series diverges.

If $p > 1$, then the series converges.

Proof:

Suppose we have $\sum_{n=1}^{\infty} \frac{1}{n^p}$

We can use the integral test.

Let $f(x) = \frac{1}{x^p}$ on $[1, \infty)$

Positivity Check:

$$\frac{1^{+}}{x^p_{+}} > 0 \text{ on } [1, \infty)$$

Decreasing Check:

$$f'(x) = -p x^{-p-1}$$

$$= \frac{-p^{-}}{x^{p+1}_{+}} < 0 \text{ on } [1, \infty)$$

Consider $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x^{-p}\,dx$

$$= \lim_{A \to \infty} \int_1^A x^{-p}\,dx$$

Case 1: $(p=1)$

$$\int_1^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{A \to \infty} \int_1^A \frac{1}{x}\,dx$$

$$= \lim_{A \to \infty} \ln|x| \Big|_1^A$$

$$= \lim_{A \to \infty} (\ln|A| - \underbrace{\ln|1|}_{0})$$

$$= \lim_{A \to \infty} \ln|A|$$

$$= \infty$$

$\therefore$ It diverges

Case 2: $(p \neq 1)$

$$\int_1^\infty x^{-p}\,dx$$

$$= \lim_{A \to \infty} \int_1^A x^{-p}\,dx$$

$$= \lim_{A \to \infty} \frac{x^{1-p}}{1-p} \Big|_1^A$$

$$= \frac{1}{1-p} \lim_{A \to \infty} \left(A^{1-p} - 1^{1-p}\right)$$

$$= \frac{1}{1-p} \lim_{A \to \infty} \left(A^{1-p} - 1\right)$$

$\therefore$ If $1-p<0$, or $p>1$, then the series convergences.

Otherwise, it divergences.

QED

6. Comparison Theorem for Series

Note: Try CT if a_n is ugly, has a p-series and/or G.S. in it.

Let $\sum a_n$, $\sum b_n$, $\sum c_n$:

Convergence Case:

If $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$ and b_n convergences, then $\sum a_n$ also convergences.

If $0 \leq c_n \leq a_n$ $\forall n \in \mathbb{N}$ and c_n diverges, then $\sum c_n$ also divergences.

Proof of Convergence Case:

Suppose 1. $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$

2. $\sum b_n$ converges

WTS: $\sum a_n$ convergences $\rightarrow \lim_{n \to \infty} S_n$ exists $\rightarrow \{S_n\}$ converges

From 1, $\{S_n\}$ is increasing.

From 2, $\{S_n\}$ is bounded

$\therefore$ By BMCT, $\{S_n\}$ converges.

$\therefore$ $\sum a_n$ convergences

7. Alternating Series / Leibniz Series (AST)

A series in the form of $\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{a_n}_{\substack{\Downarrow \\ \text{Positive Part}}} = a_1 - a_2 + a_3 \ldots \quad (a_n > 0)$

Given $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, if it satisifies both conditions, then the series converges,

1. a_n is deecreasing, I.e. $a_n > a_{n+1}$

2. If $\lim_{n \to \infty} a_n = 0$

Note: Try test 2 first. If it fails, use the divergence test. Also, this test can't check for divergence. If either test fails, it means AST is inclusive.

A series absolutely convergences if $\sum |a_n|$ convergences.
A series conditionally convergences if $\sum |a_n|$ divergences but $\sum a_n$ convergences.

8. Ratio Test (RT)
Let $\sum a_n$ be a series.
Define $p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Note, $p > 0$
If $p < 1 \Rightarrow \sum a_n$ absolutely convergences
If $p > 1 \Rightarrow \sum a_n$ divergences
If $p = 1$, the ratio test is inconclusive and use another test.

You should use this when you get factorials and complicated products.

Power Series (PS)
Let $\{C_n\} \in \mathbb{R}$, $a \in \mathbb{R}$
Suppose we have $\underbrace{\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 \ldots}$

$\Downarrow$

Power Series

C_n: n^{th} term coefficient
$C_n(x-a)^n$: General term
a: Center of Power Series

E.g. $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots x^k$

$C_n = 1$
$a = 0$

Power Series will converge for certain values of x and diverge for others.

Step 1.
Find the radius of convergence using the ratio test.

Step 2.
Check if the end points converge or diverge.

E.g. Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{(n)(4^n)}$

Step 1.
Using the ratio test, consider $\lim_{n\to\infty} \left| \frac{(C_{n+1})(x-a)^{n+1}}{(C_n)(x-a)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}(x-2)^{n+1}}{(n+1)(4^{n+1})} \div \frac{\cancel{(-1)^n}(x-2)^n}{(n)(4^n)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)\cancel{(x-2)^n}}{(n+1)(4)\cancel{(4^n)}} \cdot \frac{(n)\cancel{(4^n)}}{\cancel{(x-2)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n(x-2)}{4(n+1)} \right|$$

$$= \lim_{n\to\infty} \frac{n|x-2|}{4(n+1)}$$

$$= \frac{|x-2|}{4} \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{|x-2|}{4}$$

By R.T., we know that we get A.C. if $\frac{|x-2|}{4} < 1$

and that we get divergence if $\frac{|x-2|}{4} > 1$.

$\frac{|x-2|}{4} < 1$

$= |x-2| < 4$

$\therefore$ The radius is 4.

Step 2.

· Check if $x = a \pm r$ conv or div.

$a = 2$

$r = 4$

1. $x = a - r$

$= 2 - 4$

$= -2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-2-2)^n}{n(4^n)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{(n)(4^n)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n \cancel{(4)}^n}{(n)\cancel{(4)}^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

By P-series, $\frac{1}{n}$ div.

$\therefore$ $x = -2$ div

2. $x = a + r$

$= 2 + 4$

$= 6$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (6-\cancel{2})^n}{(n)\cancel{(4)}^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

We can use AST.

$b_n = \frac{1}{n} > 0$

$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$

$(\frac{1}{n})' = \frac{-1}{n^2} < 0$

$\therefore$ By AST, $x = 6$ conv

The final answer $= (-2, 6]$.

Properties of Conv Series

Let $\sum a_n$, $\sum b_n$ be series.

If $\sum a_n = a$ and $\sum b_n = b$, for some number a and $b \in \mathbb{R}$

Then:

1. $c \sum a_n = \sum c a_n = ca$
2. $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = a \pm b$
3. $\lim_{n \to \infty} a_n = 0$ (Vanishing Condition)

Note that $\lim_{n \to \infty} a_n = 0$ does not mean a_n convergences.

Proof of Vanishing

Suppose $\sum a_n$ conv to a, $a \in \mathbb{R}$

I.e. $\lim_{n \to \infty} S_n = a$, where $S_n = a_1 + a_2 + \ldots + a_n$

WTS: $\lim_{n \to \infty} a_n = 0$

$$\lim_{n \to \infty} a_n$$

$$= \lim_{n \to \infty} (a_1 + a_2 + \ldots + a_n) - (a_1 + a_2 + \ldots + a_{n-1})$$

$$= \lim_{n \to \infty} S_n - S_{n-1}$$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$$

$$= S - S$$

$= 0$, as wanted

QED