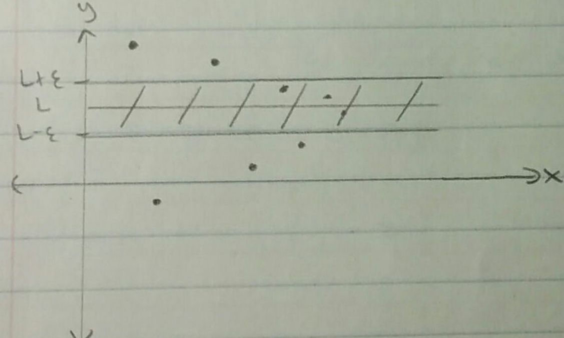
Sequences

- An infinite ordered list of numbers.
- E.g. 1, 2, 3 ...
- A sequence of real numbers is a real-valued function whose domain is $\mathbb{N}$.
- Can be denoted as 1. $\{a_n\}$
  2. $\{a_n\}_{n=1}^{\infty}$
  3. $a_n = f(n)$, where a_n is the general term

Sequences can converge or diverge.

A sequence, $\{a_n\}$, converges to a real number, $L \in \mathbb{R}$ iff $\exists L \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < \varepsilon$

Geometrically:



E.g. Prove that $\left\{ \frac{3}{\sqrt{n} + \sin^2 n} \right\}$ converges to 0

WTS: $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - 0| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

Choose $N = \frac{9}{\varepsilon^2} > 0$

Note: Choose N at the end of the process.

Suppose $n>N$

Consider $|a_n - 0|$

$$= \left| \frac{3}{\sqrt{n} + \sin^2 n} \right|$$

$$= \frac{3}{\sqrt{n} + \sin^2 n} \quad \text{By abs value properties}$$

$$\leq \frac{3}{\sqrt{n}} \quad \text{To minimize the denominator}$$

Since $n>N$, $\frac{1}{n} < \frac{1}{N}$, $\frac{3}{\sqrt{n}} < \frac{3}{\sqrt{N}}$

$$\frac{3}{\sqrt{N}} = \varepsilon$$

$$\sqrt{N} = \frac{3}{\varepsilon}$$

$$N = \frac{9}{\varepsilon^2} \quad \text{This is where you choose N.}$$

$$\frac{3}{\sqrt{N}}$$

$$= \frac{3}{\sqrt{\frac{9}{\varepsilon^2}}}$$

$$= \frac{3}{\left(\frac{3}{\varepsilon}\right)}$$

$= \varepsilon$, as wanted

QED

Sometimes, you need to make a helper function.

E.g. Prove that $a_n = \frac{n^2-2}{n^2+2n+2}$ converges.

If they don't give you L, you have to evaluate the limit of a_n to find it.

$$\lim_{n\to\infty} \frac{n^2-2}{n^2+2n+2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1-\frac{2}{n^2})}{\cancel{n^2}(1+\frac{2}{n}+\frac{2}{n^2})}$$

$$= \lim_{n\to\infty} \frac{1-\cancel{\frac{2}{n^2}}}{1+\cancel{\frac{2}{n}}+\cancel{\frac{2}{n^2}}}$$

$= 1$

$\therefore L=1$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \frac{6}{\varepsilon} > 0$

Suppose $n > N$

Consider $|a_n - 1|$

$$= \left| \frac{n^2-2}{n^2+2n+2} - 1 \right|$$

$$= \left| \frac{n^2-2-(n^2+2n+2)}{n^2+2n+2} \right|$$

$$= \left| \frac{-2n-4}{n^2+2n+2} \right|$$

$$= \left| \frac{-2(n+2)}{n^2+2n+2} \right|$$

$= \frac{2(n+2)}{n^2+2n+2}$ By abs value properties

$< \frac{2(n+2)}{n^2}$ To min the denominator

$< \frac{2(1+\frac{2}{n})}{n}$ ← If $n=0$, then $(\frac{2}{n})$ would be a problem. We can use a helper function to fix this.

Assume $n > 1$

Since $n > 1$, then $\frac{2}{n} < 2$ and $1 + \frac{2}{n} < 3$

$$\frac{2(1+\frac{2}{n})}{n}$$

$$< \frac{6}{n}$$

$$< \frac{6}{N}$$ b/c $n > N$, so $\frac{1}{n} < \frac{1}{N}$

$$\frac{6}{N} = \varepsilon$$

$$N = \frac{6}{\varepsilon}$$

However, because of our helper function, $N \geq 1$.
To satisfy both conditions, $N = \max(1, \frac{6}{\varepsilon})$.
Without loss of generality, $N = \frac{6}{\varepsilon}$.

We can ignore the $N \geq 1$ if we say that

$$\frac{6}{N}$$

$$= \frac{6}{(\frac{6}{\varepsilon})}$$

$= \varepsilon$, as wanted

If a sequence diverges, there are 2 ways to prove it.

1. Proof by Contradiction
2. Prove by infinite / negative infinite limit

E.g. of 1

Prove that $\{1+(-1)^n\}$ diverges.

Suppose $\{1+(-1)^n\}$ converges to some number, $L \in \mathbb{R}$.

We know that $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L| < \varepsilon$

Let $\varepsilon = 1$

$\therefore \exists N > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < 1$

When n is odd and if $n > N$: $|1-1-L| < 1$

$$|-L| < 1$$

$$|L| < 1$$

$$-1 < L < 1$$

When n is even and if $n > N$: $|1+1-L| < 1$

$$|2-L| < 1$$

$$|L-2| < 1$$

$$-1 < L-2 < 1$$

$$1 < L < 3$$

L is in both $(-1,1) \cap (1,3)$. But they are disjoint sets, so that's a contradiction.

$\therefore$ $\{1+(-1)^n\}$ diverges

QED

E.g. of 2

Prove $\{n^2\}$ diverges

$\lim_{n\to\infty} n^2 = \infty$

WTS: $\forall M>0, \exists N>0$ s.t. $\forall n \in \mathbb{N}$ if $n>N$ then $a_n > M$

Let $M>0$ be arbitrary.

Choose $N = \underline{\sqrt{M}} > 0$

Suppose $n>N$

Consider n^2
$> N^2$

$N^2 = M$
$N = \sqrt{M}$

N^2
$= (\sqrt{M})^2$
$= M$, as wanted

QED

Theorems

Let $\{a_n\}$ and $\{b_n\}$ be sequences.

If $\underbrace{\lim_{n\to\infty} a_n = a}$ and $\underbrace{\lim_{n\to\infty} b_n = b}$, for some real numbers a, b.

This means $\{a_n\}$ converges to a.

This means $\{b_n\}$ converges to b.

1. $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$

2. $\lim_{n\to\infty} c(a_n) = c \lim_{n\to\infty} a_n = ca$ Note: 2 is a special case of 3.

3. $\lim_{n\to\infty} (a_n \cdot b_n) = ab$

4. Every converging sequence has a unique limit.

5. Every converging sequence is bounded.
   Note, the converse in general is False.
   I.e. if $\{a_n\}$ is bounded $\not\Rightarrow$ $\{a_n\}$ converges

6. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, $b \neq 0$, $b_n \neq 0$ $\forall n \in \mathbb{N}$

Proof of 1.

Suppose $\{a_n\}$ converges to a ①

$\{b_n\}$ converges to b ②

WTS: $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|(a_n + b_n) - (a+b)| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \underline{\max(N_1, N_2)} > 0$

From ①:

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \frac{\varepsilon}{2}$

From ②:

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|b_n - b| < \frac{\varepsilon}{2}$

Suppose $n > N$

Consider $|(a_n + b_n) - (a+b)| < \varepsilon$

$= |a_n - a + b_n - b| < \varepsilon$

$\leq \underset{< \frac{\varepsilon}{2}}{|a_n - a|} + \underset{< \frac{\varepsilon}{2}}{|b_n - b|}$ (Triangle Inequality)

$< \varepsilon$, as wanted

QED

Proof of 2 and 3

Suppose $\{a_n\}$ converge to a ①
$\{b_n\}$ converge to b ②

WTS: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n b_n - ab| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

Choose $N = \underline{\max(N_1, N_2)} > 0$

From ①

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \frac{\varepsilon}{2b_n}$

From ②

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|b_n - b| < \frac{\varepsilon}{2a}$

Consider $|a_n b_n - ab|$
$= |a_n b_n - a b_n + a b_n - ab|$
$\leq |a_n - a||b_n| + |b_n - b||a|$
$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$
$\leq \varepsilon$, as wanted

Proof of 6

Suppose $\{a_n\}$ converges to a. ①
$\{b_n\}$ converges to b. ②

WTS: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \underline{\max(N_1, N_2)} > 0$

From ①

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \frac{b\varepsilon}{2}$

From ②

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|b_n - b| < \frac{(b)(b_n)(\varepsilon)}{2a}$

Suppose $n > N$

Consider $\left| \frac{a_n}{b_n} - \frac{a}{b} \right|$

$$= \left| \frac{a_n(b) - a(b_n)}{(b)(b_n)} \right|$$

$$= \left| \frac{a_n(b) - b_n(a) + (a_n b_n) - (a_n b_n)}{(b)(b_n)} \right|$$

$$= \left| \frac{-a_n(b_n - b) + b_n(a_n - a)}{(b)(b_n)} \right|$$

$$= \left| \frac{b_n(a_n - a)}{b(b_n)} - \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$= \left| \frac{(a_n - a)}{b} - \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$\leq \left| \frac{(a_n - a)}{b} \right| + \left| - \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$\leq \left| \frac{(a_n - a)}{b} \right| + \left| \frac{a_n(b_n - b)}{(b)(b_n)} \right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\leq \varepsilon$, as wanted

QED

Proof of 4

Suppose that $\{a_n\}$ converges to L_1 and L_2 and $L_1 \neq L_2$

Let $\varepsilon = \frac{|L_1 - L_2|}{2} > 0$ This is to prevent overlapping

$\therefore$

$\exists N_1 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L_1| < \frac{|L_1 - L_2|}{2}$

$\exists N_2 > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - L_2| < \frac{|L_1 - L_2|}{2}$

Choose $N = \max(N_1, N_2)$

$$
\begin{aligned}
|L_1 - L_2| &= |a_n - L_1 + a_n - L_2| \\
&\leq |a_n - L_1| + |a_n - L_2| \quad \text{By Triangle Inequality} \\
&< \left|\frac{L_1 - L_2}{2}\right| + \left|\frac{L_1 - L_2}{2}\right| \\
&< |L_1 - L_2|
\end{aligned}
$$

$\therefore$ This is a contradiction.

$\therefore$ The limit is unique.

QED

Proof of 5

Suppose $\{a_k\}$ converges to some number, $a \in \mathbb{R}$.

WTS: $\forall \varepsilon > 1$, $\exists N > 0$ s.t. $\forall n \in \mathbb{N}$ if $n > N$ then $|a_n - a| < \varepsilon$

Let $\varepsilon = 1$

Suppose $n > N$

$|a_n - a| < 1$

By triangle inequality, $|a_n| - |a| < 1$

$|a_n| < |a| + 1$

Let $M = \max\{|a| + 1, |a_1|, |a_2| \ldots |a_n|\}$

Then, we have $|a_n| \leq M \; \forall n \in \mathbb{N}$, as wanted

QED

Recursive Sequences

Recursive sequences are sequences such that if there is some index k, so that for $k > K$, the value of a_k is determined by $a_1, a_2, a_3 \ldots a_{k-1}$.

E.g.

$a_1 = \sqrt{6}$

$a_{n+1} = \sqrt{6 + a_n}$

In this case, $k = 1$ and for $k > 1$, the value of a_k is determined by $a_1, a_2, \ldots a_{k-1}$.

For recursive sequences, use BMCT instead of epsilon proofs.

Bounded Monotone Convergence Theorem (BMCT)

If $\{a_n\}$ is $\underbrace{\text{bounded and monotone}}$, then $\{a_n\}$ converges.

$\downarrow$

1. Bounded above and strictly increasing

OR

2. Bounded below and strictly decreasing.

Proof of BMCT (Bounded above and strictly increasing.)

Suppose that $\{a_n\}$ is 1. Bounded above

2. Strictly increasing

WTS: $\{a_n\}$ converges $\Leftrightarrow \exists L \in \mathbb{R}, \forall \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$, if $n > N$ then $|a_n - L| < \varepsilon$

Define $A = \{a_n \mid n \in \mathbb{N}\} \in \mathbb{R}$

Note: $A \neq \emptyset$ because $a_1 \in A$

Furthermore, A is bounded above by Statement 1.

By the Completeness Axiom, $\sup(A) = \alpha \in \mathbb{R}$ exists.

Choose $L = \alpha$

Let $\varepsilon > 0$ be arbitrary

Choose $N =$ ______ > 0 — Let's pick N's properties instead.

$N \in \mathbb{N}$ s.t. $\alpha - \varepsilon < a_N$

Suppose $n > N$

$\therefore \alpha - \varepsilon < a_N \leq a_n \leq \alpha < \alpha + \varepsilon$

By choice of N — Because $n > N$ and of statement 2 — Because $\alpha = \sup\{a_n\}$ — because $\varepsilon > 0$

$\Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$

$= -\varepsilon < a_n - \alpha < \varepsilon$

$= |a_n - \alpha| < \varepsilon$

$= |a_n - L| < \varepsilon$, as wanted

QED

Proof of BMCT (Bounded below and strictly decreasing)

Suppose that $\{a_n\}$ is 1. Bounded Below
2. Strictly Decreasing

Define $A = \{a_n \mid n \in \mathbb{N}\} \in \mathbb{R}$

Note: $A \neq 0$ because $a_1 \in A$

Moreover, A is bounded below by statement 1.

$\therefore$ By Conpletness Axiom, $\inf(A) = \alpha \in \mathbb{R}$ exists

Choose $L = \alpha$

Let $\varepsilon > 0$ be arbitrary

Choose $N = \underline{\qquad\quad} > 0$ Let's pick N's properties instead.

$N \in \mathbb{N}$ s.t. $a_N < \alpha + \varepsilon$.

Suppose $n > N$

So $\alpha - \varepsilon < \alpha \leq a_n \leq a_N < \alpha + \varepsilon$

B/C $\varepsilon > 0$; B/C a_n is bounded below by α. ; B/c $n > N$ and a_n is bounded below. ; By definition of N

$\Rightarrow \alpha - \varepsilon < a_N < \alpha + \varepsilon$

$= -\varepsilon < a_n - \alpha < \varepsilon$

$= |a_n - \alpha| < \varepsilon$

$= |a_n - L| < \varepsilon$, as wanted

QED

E.g. Prove that the sequence $a_1 = \sqrt{6}$, $a_{n+1} = \sqrt{6 + a_n}$ if $n > 1$ converges.

1. Have to prove that the sequence is bounded.

Check if it's bounded above or below by rough work first.

$a_1 = \sqrt{6}$
$a_2 = \sqrt{6+\sqrt{6}}$
$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$

They are increasing, so this sequence is bounded above.

Use induction to prove that the sequence is bounded above.

First, use rough work to find which number is bounding the sequence.

$a_1 = \sqrt{6} < \sqrt{9} = 3$ b/c $0 < 6 < 9$
$a_2 = \sqrt{6+\sqrt{6}} < \sqrt{6+3} = 3$
$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}} < \sqrt{6+3} = 3$

$\therefore$ The sequence is bounded by 3.

Induction $\longrightarrow$ ① Base Case:

We know by definition of $\{a_n\}$ that $a_1 = \sqrt{6} < \sqrt{9} = 3$.
This is because $0 < 6 < 9$.

② Induction Hypothesis:

$\forall k \in \mathbb{N}$, if $a_k < 3$ then $a_{k+1} < 3$.
OR $\forall k \in \mathbb{N}$, $a_k < 3 \Rightarrow a_{k+1} < 3$

③ Induction Step:

Let $k \in \mathbb{N}$ be arbitrary.
Suppose $a_k < 3$ (By I.H)

Consider $a_{k+1} = \sqrt{6+\underbrace{a_k}_{<3}}$ (By def of $\{a_n\}$, $k \in \mathbb{N}$) (By I.H)

$< \sqrt{6+3}$

$< \sqrt{9}$

< 3, as wanted

QED

2. Have to prove $\{a_n\}$ is strictly increasing.

3 ways to prove a sequence is increasing.

a) Difference test: $a_{k+1} - a_k \geq 0 \ \forall k \geq 1$

b) Ratio Test: If all terms are positive, and $\frac{a_{k+1}}{a_k} \geq 1 \ \forall k \geq 1$

c) Derivative Test: $a'(x) \geq 0 \ \forall x > 1$, given that $a(x)$ is differentiable on $[1, \infty)$ and $a_k = a(k) \ \forall k \geq 1$.

Note: To prove that a sequence is strictly decreasing, take the opposite of the 3 ways.

Consider $(a_n)^2 - (a_{n+1})^2$

$= (a_n)^2 - (\sqrt{6+a_n})^2$ By def of $\{a_n\}$

$= (a_n)^2 - a_n - 6$

$= (a_n - 3)(a_n + 2)$

Since $a_n < 3$, $a_n - 3 < 0$

$a_n + 2 > 0$

So, $(a_n - 3)(a_n + 2) < 0$

$\therefore \forall n \in \mathbb{N}$, $(a_n)^2 - (a_{n+1})^2 < 0 \iff 0 < (a_n)^2 < (a_{n+1})^2$

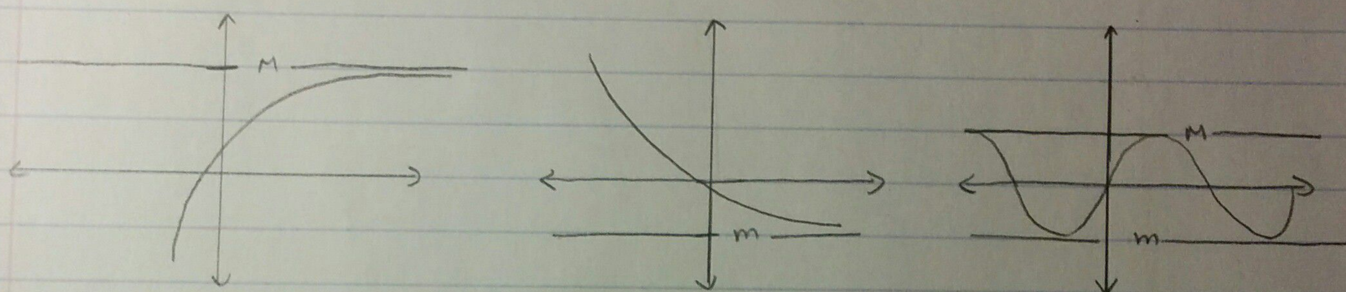
$\therefore a_n < a_{n+1}$, as wanted

QED

Bounding and Monotone

A sequence $\{a_n\}$ is:

1. Bounded above $\iff \exists M \in \mathbb{R}$ s.t. $a_n \leq M \ \forall n \in \mathbb{N}$
2. Bounded below $\iff \exists m \in \mathbb{R}$ s.t. $a_n \geq m \ \forall n \in \mathbb{N}$
3. Bounded $\iff \{a_n\}$ is bounded above or below.
   $\iff \exists C \in \mathbb{R}, C > 0$ s.t. $|a_n| \leq C$



Bounded above　　　　Bounded Below　　　　Bounded

A sequence $\{a_n\}$ is:

1. Increasing $\iff a_n \leq a_{n+1} \ \forall n \in \mathbb{N}$
2. Decreasing $\iff a_n \geq a_{n+1} \ \forall n \in \mathbb{N}$
3. Strictly Increasing $\iff a_n < a_{n+1} \ \forall n \in \mathbb{N}$
4. Strictly Decreasing $\iff a_n > a_{n+1} \ \forall n \in \mathbb{N}$
5. Montone $\iff \{a_n\}$ is either strictly increasing or strictly deereasing