II

Week 9 Notes

Proof of Trig Derivatives

1. $(\sin x)' = \cos x$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h\to 0} \frac{\sin h \cos x}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\sin x(\cos h - 1)}{h}}_{\to \sin x(0)} + \lim_{h\to 0} \underbrace{\cos x\left(\frac{\sin h}{h}\right)}_{\to \cos x(1)}$$

$$= \cos x$$

2. $(\cos x)' = -\sin x$

$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \sin x$$

$$= -\sin x$$

3. $(\tan x)' = \sec^2 x$

$$(\tan x)' = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x(\cos(x+h))}{h(\cos x)(\cos(x+h))}$$

$$= \lim_{h\to 0} \frac{\cos x(\sin x \cos h + \cos x \sin h) - \sin x(\cos x \cos h - \sin x \sin h)}{h(\cos x)(\cos(x+h))}$$

$$= \lim_{h\to 0} \frac{\cos^2 x \sin h + \sin^2 x \sin h}{h(\cos x)(\cos(x+h))}$$

$$= \lim_{h\to 0} \frac{\sin h(\cos^2 x + \sin^2 x)}{\cos x(\cos(x+h))h}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

4. $(\sec x)' = \sec x \tan x$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)'$$

$$= \frac{-(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \sec x \cdot \tan x$$

5. $(\csc x)' = -\csc x \cdot \cot x$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)'$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\csc x \cdot \cot x$$

6. $(\cot x)' = -\csc^2(x)$

$$(\cot x)' = \left(\frac{1}{\tan x}\right)'$$

$$= \frac{-\sec^2 x}{\tan^2 x}$$

$$= \frac{-1}{\cos^2 x}$$

$$= -\csc^2 x$$

Proof of Inverse Trig Functions

1. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

$y = \arcsin(x)$

$x = \sin y$

$x' = (\sin y)'$

$1 = \cos y \cdot y'$

$y' = \dfrac{1}{\cos y}$

$= \dfrac{1}{\sqrt{1-\sin^2 y}} \longleftarrow \cos x = \sqrt{1-\sin^2 x}$

$= \dfrac{1}{\sqrt{1-x^2}}$

2. $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}$

$y = \arccos x$

$x = \cos y$

$x' = (\cos y)'$

$1 = -\sin(y) \cdot y'$

$y' = \dfrac{-1}{\sin y}$

$= \dfrac{-1}{\sqrt{1-x^2}}$

3. $(\arctan x)' = \dfrac{1}{1+x^2}$

$y = \arctan x$

$x = \tan y$

$x' = (\tan y)'$

$1 = \sec^2 y \cdot y'$

$y' = \dfrac{1}{\sec^2 y}$

$= \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$

4. $(\operatorname{arc}\cot x)' = \dfrac{-1}{1+x^2}$

$y = \operatorname{arc}\cot x$

$x = \cot y$

$x' = (\cot y)'$

$1 = -\csc^2 y \cdot y'$

$y' = \dfrac{-1}{\csc^2 y}$

$= \dfrac{-1}{1+\cot^2 y}$

$= \dfrac{-1}{1+x^2}$

5. $(\operatorname{arc}\sec x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

$y = \operatorname{arc}\sec x$

$x = \sec y$

$x' = (\sec y)'$

$1 = \sec y \cdot \tan y \cdot y'$

$y' = \dfrac{1}{\sec y \cdot \tan y}$

$= \dfrac{1}{|x|\sqrt{x^2-1}}$

6. $(\operatorname{arc}\csc x)' = \dfrac{-1}{|x|\sqrt{x^2-1}}$

$y = \operatorname{arc}\csc x$

$x = \csc y$

$x' = (\csc y)'$

$1 = -\csc y \cdot \cot y \cdot y'$

$y' = \dfrac{-1}{\csc y \cdot \cot y}$

$= \dfrac{-1}{|x|\sqrt{x^2-1}}$

Hyperbolic Identities and Proofs

1. $\sinh(t) = \dfrac{e^{t} - e^{-t}}{2}$

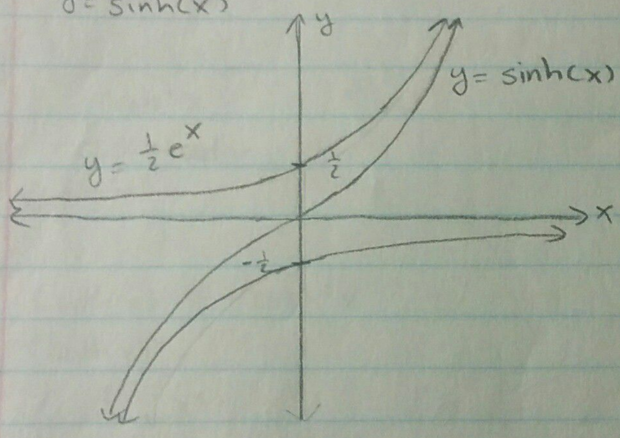
2. $\cosh(t) = \dfrac{e^{t} + e^{-t}}{2}$

3. $\tanh(t) = \dfrac{\sinh(t)}{\cosh(t)}$
   $= \dfrac{e^{t} - e^{-t}}{e^{t} + e^{-t}}$

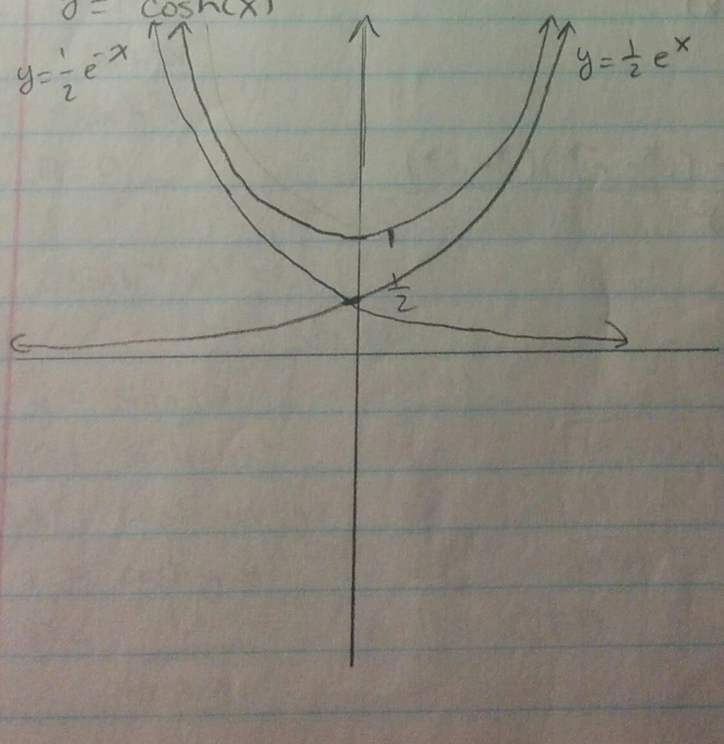
4. $\cosh^2 t - \sinh^2 t = 1$

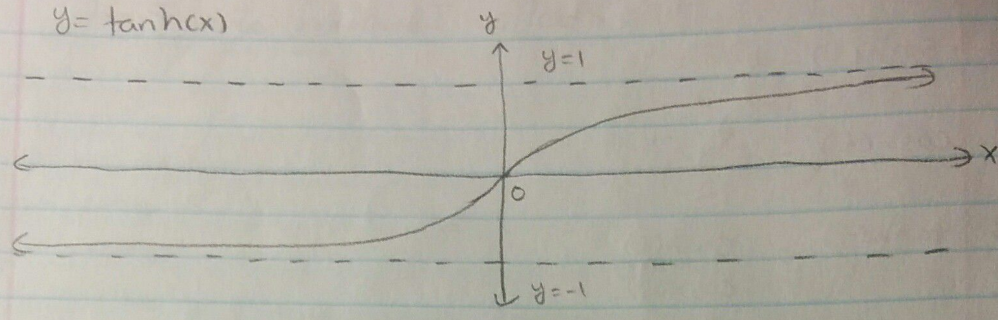
Graph of hyperbolic functions

$y = \sinh(x)$



$y = \cosh(x)$





Prove

1. $(\sinh x)' = \cosh x$

$\left[\left(\frac{1}{2}\right)(e^{x} - e^{-x})\right]'$

$= \frac{1}{2}(e^{x} + e^{-x})$

$= \cosh(x)$

QED

2. $(\cosh x)' = \sinh x$

$\left[\frac{1}{2}(e^{x} + e^{-x})\right]'$

$= \frac{1}{2}(e^{x} - e^{-x})$

$= \sinh(x)$

QED

3. $(\tanh x)' = (\operatorname{sech}^2 x)$

$\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)'$

$= \frac{(e^{x} + e^{-x})(e^{x} + e^{-x}) - (e^{x} - e^{-x})(e^{x} - e^{-x})}{(e^{x} + e^{-x})^2}$

$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$= \frac{1}{\cosh^2 x}$

$= \operatorname{sech}^2 x$

4. $(\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x$

$(\operatorname{sech} x)'$

$= \left(\frac{1}{\cosh x}\right)'$

$= \left(\frac{-\sin x}{\cosh^2 x}\right)$

$= -\operatorname{sech} x \cdot \tanh x$

QED

5. $(\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x$

$(\operatorname{csch} x)'$

$= \left(\frac{1}{\sinh x}\right)'$

$= \frac{-\cosh x}{\sinh^2 x}$

$= -\operatorname{csch} x \cdot \coth x$

QED

6. $(\coth x)' = -\operatorname{csch}^2 x$

$(\coth x)'$

$= \left(\frac{1}{\tanh x}\right)'$

$= \frac{-\operatorname{sech}^2 x}{\tanh^2 x}$

$= -\operatorname{csch}^2 x$

QED

7. $(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$

$y = \sinh^{-1} x$

$x = \sinh(y)$

$x' = \cosh(y) \cdot y'$

$1 = \cosh y \cdot y'$

$y' = \frac{1}{\cosh y}$

$y' = \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1} x)}}$

$= \frac{1}{\sqrt{x^2+1}}$

QED

8. $(\cosh^{-1} x)' = \dfrac{1}{\sqrt{x^2-1}}$

$y = \cosh^{-1} x$

$x = \cosh y$

$x' = (\cosh y)'$

$1 = \sinh y \cdot y'$

$y' = \dfrac{1}{\sinh y}$

$= \dfrac{1}{\sqrt{\sinh^2(\cosh^{-1} x) - 1}}$

$= \dfrac{1}{\sqrt{x^2-1}}$

9. $(\tanh^{-1} x)' = \dfrac{1}{1-x^2}$

$y = \tanh^{-1} x$

$x = \tanh y$

$x' = (\tanh y)'$

$1 = \operatorname{sech}^2 y \cdot y'$

$y' = \dfrac{1}{\operatorname{sech}^2 y}$

$= \dfrac{1}{\operatorname{sech}^2(\tanh^{-1} x)}$

$= \dfrac{1}{1-x^2}$

10. $(\operatorname{sech}^{-1} x)' = \dfrac{-1}{x\sqrt{1-x^2}}$

$y = \operatorname{sech}^{-1} x$

$x = \operatorname{sech} y$

$x' = (\operatorname{sech} y)'$

$1 = -\operatorname{sech} y \cdot \tanh y \cdot y'$

$y' = \dfrac{-1}{\operatorname{sech} y \cdot \tanh y}$

$= \dfrac{-1}{\operatorname{sech}(\operatorname{sech}^{-1} x) \cdot \tanh(\operatorname{sech}^{-1} x)}$

$y' = \dfrac{-1}{x\sqrt{1-x^2}}$

11 $(\operatorname{csch}^{-1} x)' = \dfrac{-1}{|x|\sqrt{1+x^2}}$

$y = \operatorname{csch}^{-1} x$

$x = \operatorname{csch} y$

$x' = (\operatorname{csch} y)'$

$1 = -\operatorname{csch} y \cdot \coth y \cdot y'$

$y' = \dfrac{-1}{\operatorname{csch} y \cdot \coth y}$

$= \dfrac{-1}{\operatorname{csch}(\operatorname{csch}^{-1} x) \cdot \coth(\operatorname{csch}^{-1} x)}$

$= \dfrac{-1}{|x|\sqrt{1+x^2}}$

12. $(\coth^{-1} x)' = \dfrac{1}{1-x^2}$

$y = \coth^{-1} x$

$x = \coth y$

$x' = (\coth y)'$

$1 = -\operatorname{csch}^2 y \cdot y'$

$y' = \dfrac{-1}{\operatorname{csch}^2 y}$

$= \dfrac{-1}{\operatorname{csch}^2(\coth^{-1} x)}$

$= \dfrac{-1}{x^2-1} = \dfrac{1}{1-x^2}$

13. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ for any x

let $y = \sinh^{-1} x$

$x = \sinh y$

$= \dfrac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$= e^{\sinh^{-1} x} - e^{-\sinh^{-1} x}$

$2xe^{\sinh^{-1} x} = (e^{\sinh^{-1} x} - e^{-\sinh^{-1} x})e^{\sinh^{-1} x}$

$= (e^{\sinh^{-1} x})^2 - 1$

Let u be $e^{\sinh^{-1}x}$

$0 = u^2 - 2xu - 1$

$u = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{2x \pm \sqrt{4x^2 + 4}}{2}$

$= x \pm \sqrt{x^2+1}$

However, since $x - \sqrt{x^2+1} < 0$, we reject this case

$\therefore \ u = x + \sqrt{x^2+1}$

$e^{\sinh^{-1}x} = x + \sqrt{x^2+1}$

$\ln(u) = \ln(x + \sqrt{x^2+1})$

$\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$

14. $\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$ for $x \geq 1$ (Proof is similar to 13)

Let $y = \cosh^{-1}x$

$x = \cosh y$

$= \dfrac{e^y + e^{-y}}{2}$

$2x = e^{\cosh^{-1}x} + e^{-\cosh^{-1}x}$

$2xe^{\cosh^{-1}x} = (e^{\cosh^{-1}x} + e^{-\cosh^{-1}x})e^{\cosh^{-1}x}$

$= (e^{\cosh^{-1}x})^2 + 1$

Let $u = e^{\cosh^{-1}x}$

$0 = u^2 - 2xu + 1$

$u = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$

$= x \pm \sqrt{x^2-1}$

Since $x - \sqrt{x^2-1} < 0$, we reject this case

$u = x + \sqrt{x^2-1}$

$\ln(u) = \ln(x + \sqrt{x^2-1})$

$\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$

15 $\tanh^{-1} x = \frac{1}{2}\left(\ln\left(\frac{1+x}{1-x}\right)\right)$ for $-1<x<1$

Let $y = \tanh^{-1} x$

$$x = \tanh y$$
$$= \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$
$$x(e^{y}+e^{-y}) = e^{y} - e^{-y}$$
$$xe^{y} + xe^{-y} = e^{y} - e^{-y}$$
$$e^{y}(xe^{y} + xe^{-y}) = e^{y}(e^{y} - e^{-y})$$
$$xe^{2y} + x = e^{2y} - 1$$
$$0 = e^{2y} - xe^{2y} - x - 1$$
$$= e^{2y}(1-x) - x - 1$$
$$x+1 = e^{2y}(1-x)$$
$$e^{2y} = \frac{x+1}{1-x}$$
$$\ln(e^{2y}) = \ln\left(\frac{x+1}{1-x}\right)$$
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Side Notes

- When doing proofs for inverse trig or inverse hyperbolic functions, start out with $y =$ (inverse trig) x
  $x =$ trig (y).
- trig func (trig inverse func x) $= x$
  Eg. $\sin(\sin^{-1} x) = x$
  BUT! trig inverse func (trig func) $\neq x$
  $\sin^{-1}(\sin x) \neq x$

- Principle Functions:
  $\sin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
  $\cos(x) \in [0, \pi]$
  $\tan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$
  $\csc(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $x \neq 0$
  $\sec(x) \in (0, \pi)$ $x \neq \frac{\pi}{2}$
  $\cot(x) \in (0, \pi)$

Summary of Derivatives

1. $\frac{d}{dx} c = 0$, c is a constant
2. $\frac{d}{dx} x = 1$
3. $\frac{d}{dx} cx = c$, c is a constant
4. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) \cdot f(x)g'(x)$
6. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
7. $[(f(x))^n]' = n(f(x))^{n-1} \cdot f'(x)$
8. $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
9. If $f(x) = a^x$, for any $a>0$, $a \neq 1$, then $f'(x) = a^x \cdot \ln a \cdot x'$
10. $(e^x)' = e^x \cdot x'$
11. If $f(x) = \log_b x$, for any $b>0$, $b \neq 1$ then $f'(x) = \frac{1}{x \ln a} \cdot x'$
12. $f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x} \cdot x'$
13. $f(x) = \ln|x| \rightarrow f'(x) = \frac{x'}{x}$ ✱ Special Case

14. $(\sin x)' = \cos x \cdot x'$	$(\sinh(x))' = \cosh(x) \cdot x'$
15. $(\cos x)' = -\sin x \cdot x'$	$(\cosh(x))' = \sinh(x) \cdot x'$
16. $(\tan x)' = \sec^2 x \cdot x'$	$(\tanh x)' = \operatorname{sech}^2 x \cdot x'$
17. $(\sec x)' = \sec x \cdot \tan x \cdot x'$	$(\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x \cdot x'$
18. $(\csc x)' = -\csc x \cdot \cot x \cdot x'$	$(\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x \cdot x'$
19. $(\cot x)' = -\csc^2 x \cdot x'$	$(\coth x)' = -\operatorname{csch}^2 x \cdot x'$
20. $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \cdot x'$	$(\sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}} \cdot x'$
21. $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}} \cdot x'$	$(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}} \cdot x'$
22. $(\tan^{-1} x)' = \frac{1}{1+x^2} \cdot x'$	$(\tanh^{-1} x)' = \frac{1}{1-x^2} \cdot x'$
23. $(\sec^{-1} x)' = \frac{1}{\lvert x \rvert \sqrt{x^2-1}} \cdot x'$	$(\operatorname{sech}^{-1} x)' = \frac{-1}{x\sqrt{1-x^2}} \cdot x'$
24. $(\csc^{-1} x)' = \frac{-1}{\lvert x \rvert \sqrt{x^2-1}} \cdot x'$	$(\operatorname{csch}^{-1} x)' = \frac{-1}{\lvert x \rvert \sqrt{1+x^2}} \cdot x'$
25. $(\cot^{-1} x) = \frac{-1}{1+x^2} \cdot x'$	$(\coth^{-1} x)' = \frac{1}{1-x^2} \cdot x'$