Proving Derivative Rules

1. Constant Multiple Rule

Prove $(rf)'(x) = rf'(x)$

$$\lim_{h\to 0} \frac{r(f(x+h)) - r(f(x))}{h}$$

$$= \lim_{h\to 0} \frac{r(f(x+h) - f(x))}{h}$$

$$= r \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= rf'(x)$$

2. Sum Rule

Prove $(f+g)'(x) = f'(x) + g'(x)$

$$\lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

3. Difference Rule

Prove $(f-g)'(x) = f'(x) - g(x)$

$$\lim_{h\to 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x)$$

4. Product Rule

Prove $(f \cdot g)'(x) = f'(x) g(x) + f(x) g'(x)$

$$\lim_{h \to \infty} \frac{f(x+h) g(x+h) - f(x) g(x) - f(x) g(x+h) + f(x) g(x+h)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)] g(x+h) + [g(x+h) - g(x)] f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} f(x)$$

$$= f'(x) g(x) + f(x) g'(x)$$

5. Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = (f(x) \cdot [g(x)]^{-1})'$$

$$= \frac{f'(x)}{g(x)} + \frac{f(x)(-1)(g'(x))}{(g(x))^2}$$

$$= \frac{f'(x) g(x) - f(x) g'(x)}{(g(x))^2}$$

6. Show that $(e^x)' = e^x$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^h e^x - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x (eh-1)}{h}$$

$$= \lim_{h \to 0} e^x + \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x$$

②

7. Prove that if $f(x) = a^x$, then $f'(a) = a^x \ln a$

$a^x = e^{x\ln a}$

$(a^x)' = (e^{x\ln a})'$

$= e^{x\ln a}(x\ln a)'$

$= a^x \ln a$

8. Prove that if $f(x) = \log_a x$, for any $a > 0$, $a \neq 1$, then $f'(x) = \frac{1}{x\ln a}$

$x = a^{\log_a x}$

$x' = (a^{\log_a x})'$

$= a^{\log_a x}(\ln a)(\log_a x)'$

$1 = x\ln a(\log_a x)'$

$(\log_a x)' = \frac{1}{x\ln a}$

9. Prove that if $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$

$e^{f(x)} = e^{\ln x}$

$(e^{f(x)})' = (e^{\ln x})'$

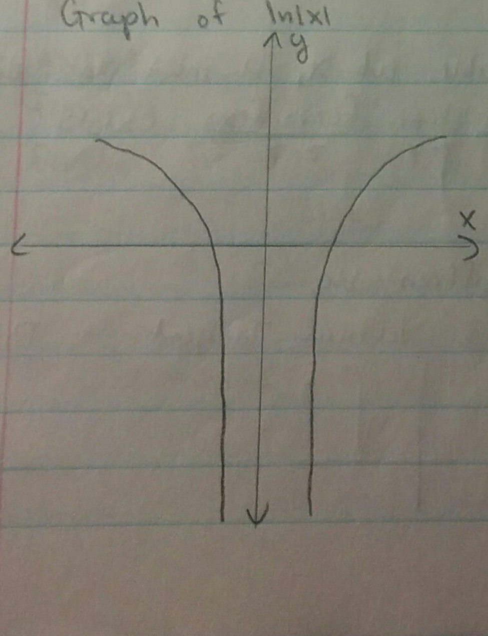
$e^{f(x)} \cdot f'(x) = x'$

$= 1$

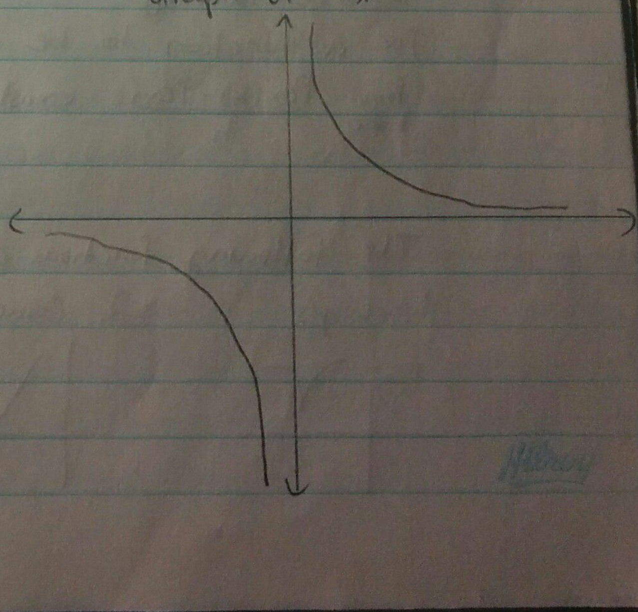
$f'(x) = \frac{1}{x}$

10 Prove if $f(x) = \ln|x|$, then $f'(x) = \frac{1}{x}$

Graph of $\ln|x|$



Graph of $\frac{1}{x}$



11. Chain Rule Proof

Prove $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \left(\frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right)$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x)$$

Equation of a tangent line

$f(x) = f(a) + f'(a)(x-a)$

Eg. Find the eqn of the tangent line to x^2+2x+1 at $x=2$.

$a = 2$

$f(a) = (2)^2 + 2(2) + 1$

$= 4+4+1$

$= 9$

$f'(a) = 2x+2$

$= 2(2)+2$

$= 6$

$f(x) = 9 + 6(x-2)$

$= 6x - 12 + 9$

$= 6x - 3$

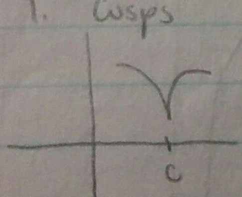
Diffrentiability:

For a function to be diffrentiable at x, it must pass either rule:

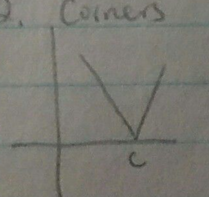
$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists <u>OR</u> $\lim_{z \to x} \frac{f(z) - f(x)}{z - x}$ exists

The following functions are not diffrentiable

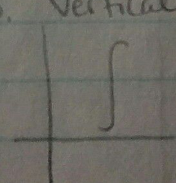
1. Cusps



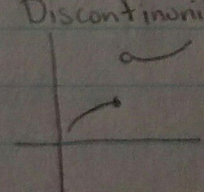
2. Corners



3. Vertical Tangent



4. Discontinuity



12. Proof of Power Rule

Prove $f'(x) = n(x^{n-1})$

$(a+b)^n = a^n + na^{n-1}b \ldots + b^n$

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{x^n + nx^{n-1}h \ldots + h^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h \ldots + h^n}{h}$$

$$= nx^{n-1}$$

13. Proof of Constant Function

Show $f'(x) = 0$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h}$$

$$= 0$$

Linearization and Newton's Method

1. Linearization
   1. $f(x) = f(a) + f'(a)(x-a)$
   2. $f(a+\Delta x) = f(a) + f'(a)\Delta x$

E.g. Find the linear approximation of $\sqrt{16.1}$

$a = 16$

$\Delta x = 0.1$

$f(x) = \sqrt{x}$

$$f(a+\Delta x) = \sqrt{16} + \frac{1}{2\sqrt{16}}(0.1)$$

$$= 4 + \frac{0.1}{8}$$

$$= 4.0125$$

$\Delta y = f(x+\Delta x) - f(x)$

$dy = f'(a)\,dx$

E.g. Find Δy and dy for the function $f(x) = \frac{1}{x+2}$ when $x=1$, $\Delta x = 0.01$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= \frac{1}{1.01+2} - \frac{1}{1+2}$$
$$= \frac{1}{3.01} - \frac{1}{3}$$
$$= -0.0011074$$

$$dy = f'(a)\,dx$$
$$= \frac{-1}{(x+2)^2}(0.01) \quad \leftarrow \Delta x = dx$$
$$= \frac{-0.01}{(1+2)^2}$$
$$= \frac{-0.01}{9}$$
$$\approx -0.0011$$

2. Newton's Method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

E.g. Find at least 1 root of $x^3 - 2x - 5 = 0$ with the accuracy up to 3 decimal places.

$f(0) = -5$

$$f(3) = 3^3 - 2(3) - 5$$
$$= 27 - 6 - 5$$
$$= 16$$

$\therefore$ The root is between $x=0$ and $x=3$

Let $x=2$ be x_0

$$x_1 = 2 - \left(\frac{-1}{10}\right) \qquad x_2 = 2.1 - \left(\frac{0.061}{11.23}\right)$$
$$= 2.1 \qquad\qquad = 2.095$$

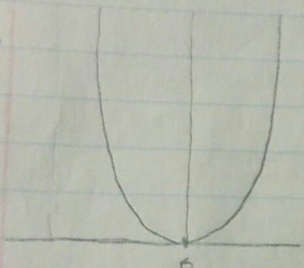
$\therefore$ The root is approx 2.095

Eqn of secant line: $y - f(a) = \frac{f(a+h) - f(a)}{h}(x-a)$

Derivative = Slope of tangent = Instanteous Rate of Change

Relationships Btwn $f(x)$ and $f'(x)$:
1. For each x, the slope of $f(x)$ is the height of $f'(x)$
2. Where $f(x)$ has a horizontal tangent, $f'(x)$ has a root
3. Whenever $f(x)$ is increasing, $f'(x) > 0$
4. Whenever $f(x)$ is decreasing, $f'(x) < 0$
5. Whenever $f(x)$ has a steep slope, $f'(x)$ has a large magnitude
6. Whenever $f(x)$ has a shallow slope, $f'(x)$ has a small magnitude

E.g.

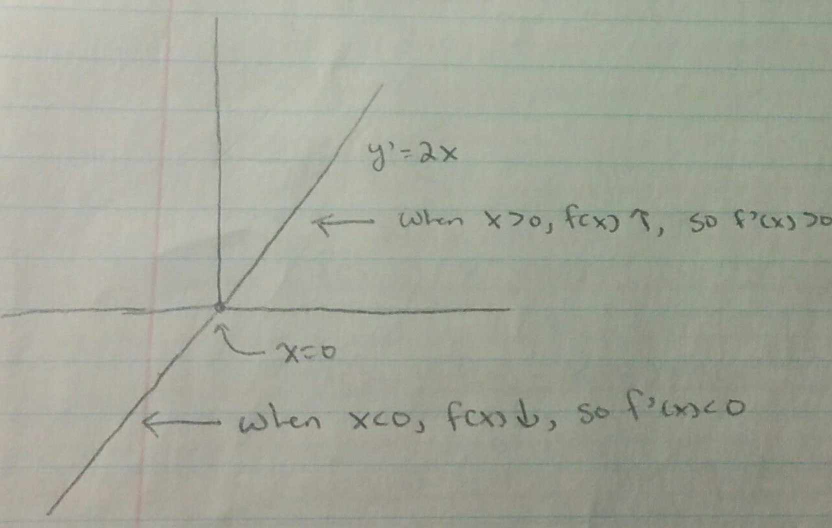


$y = x^2$

At $x = 0$, there is a horizontal tangent

If $x < 0$, y is decreasing

If $x > 0$, y is increasing



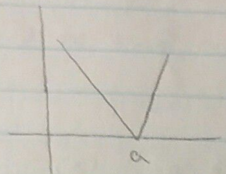
Left Derivative

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

Right Derivative

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

When doing questions with piecewise functions or



, you have to check continuity and if left derivative = right derivative.

Differentiability implies continuity but continuity does NOT imply differentiability.