* Prove $\lim_{x \to \infty} \left(1+\frac{k}{x}\right)^x = e^k$

$$e^{x \ln(1+\frac{k}{x})} = \left(1+\frac{k}{x}\right)^x$$

$$\lim_{x \to \infty} \left(1+\frac{k}{x}\right)^x = \lim_{x \to \infty} e^{x \ln(1+\frac{k}{x})}$$

$$= e^{\lim_{x \to \infty} x \ln(1+\frac{k}{x})}$$

$$= e^{\lim_{x \to \infty} \frac{\ln(1+\frac{k}{x})}{\frac{1}{x}}}$$

$$= e^{\lim_{x \to \infty} \frac{k}{1+\frac{k}{x}}}$$

$$= e^k$$

QED

* Prove $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

let $y = (1+x)^{\frac{1}{x}}$

$$\ln(y) = \ln\left((1+x)^{\frac{1}{x}}\right)$$

$$= \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \left(\frac{1}{x}\right)(\ln(1+x))$$

Using l'hôpital's Rule, we get

$$= \lim_{x \to 0} \frac{1}{1+x}$$

$$= 1$$

$$\ln(y) = 1$$

$$y = e$$

$$\therefore \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

QED

Bisectional Method / Method of Bisections

- Used to find approx value of roots
- Recall that according to the IVT, a function can only change signs at roots or discontinuities.
- That means, if you know a function is cont with closed interval $[a,b]$, and you know that the function changes signs, there must be a root.

Eg.

$f(x) = x^3 + x^2 + 3x + 5$

Show that $f(x)$ has a root in $[-2,0]$.

$f(0) = 5$

$f(-2) = -5$

Since $f(x)$ is a polynomial, it is cont in $[-2,0]$.
Since $f(x)$ changes sign between 0 and -2, we know there must be a root between 0 and -2.

To find the root, we'll find the midpoint of 0 and -2 and see if its positive or negative.

$f(-1) = 2$

This means the root is between -1 and -2

$f(-1.5) = -0.625$

This means the root is between -1 and -1.5

Since the numbers are getting harder to calculate, I will stop here.

P.S. The root is -1.4, which concludes that my work is right.

Prove $\lim_{x \to c} k = k$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |k-k| < \varepsilon$

Since $k-k=0$ and $\varepsilon > 0$, this statement is always true. QED

Prove $\lim_{x \to c} x = c$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |x-c| < \varepsilon$

Choose $d = \varepsilon$

$|x-c| < \varepsilon$

But $|x-c| < d$

$d = \varepsilon$

$\varepsilon = \varepsilon$

QED

Prove $\lim_{x \to c} mx+b = mc+b$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |mx+b-mc-b| < \varepsilon$

$|mx+b-mc-b|$

$= |mx-mc|$

$= m|x-c|$

$= md$

$md = \varepsilon$

$d = \varepsilon/m$

$|mx+b-mc-b| < \varepsilon$

$|mx-mc| < \varepsilon$

$m|x-c| < \varepsilon$

$m(\varepsilon/m)$

$= \varepsilon$

QED

Summary of Limit Laws

1. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$

2. $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

3. $\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \lim_{x \to c} g(x) \neq 0$

5. $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}, \quad M \neq 0$

6. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$, n is a positive int

7. $\lim_{x \to c} \sqrt[n]{f(x)} = \left[\sqrt[n]{\lim_{x \to c} f(x)}\right]$, n is a positive int

8. $\lim_{x \to c} k = k$

9. $\lim_{x \to c} x = c$

10. $\lim_{x \to c} (mx+b) = \lim_{x \to c} (mc+b)$

11. $\lim_{x \to c} Ax^k = Ac^k$

12. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

13. $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

14 $\lim_{x \to \infty} \left(1+\frac{k}{x}\right)^x = e^k$

15 $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

Indeterminate Forms of Limits:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

Non-Indeterminate Forms of Limits:

a) A limit in any of these forms is equal to 0

$\frac{1}{\infty}$, $\frac{0}{\infty}$, $\frac{0}{1}$, 0^∞, 0^1

b) A limit in any of these forms is equal to infinity.

$\frac{1}{0^+}$, $\frac{\infty}{0^+}$, $\frac{\infty}{1}$, $\infty + \infty$, $\infty \cdot \infty$, ∞^∞, ∞^1

Tips on evaluating limits:

1. Most trig limits can be solved using S.T.

E.g.

Evaluate $\lim\limits_{x \to 0} x\sin(\frac{1}{x})$

We know that $\sin(x) \in (-1, 1)$

$\therefore \sin(\frac{1}{x}) \in -1, 1$

$-1 \leq \sin(\frac{1}{x}) \leq 1$

$-x \leq x\sin(\frac{1}{x}) \leq x$

$\lim\limits_{x \to 0} -x = 0$ $\quad \lim\limits_{x \to 0} x = 0$

$\therefore$ By S.T., $\lim\limits_{x \to 0} x\sin(\frac{1}{x}) = 0$

Lemma proof for IVT

1. If $f(x)$ is cont on closed interval $[a,b]$ and $f(a)<0<f(b)$ then there is a number c, such that if $c\in(a,b)$ then $f(c)=0$.

Proof:

Let $f(a)<0<f(b)$

Since $f(a)<0$ and $f(x)$ is cont on $[a,b]$, then there exists a number, m, such that $f(x)$ is negative on $[a,m)$. Set $[a,m)$ is bounded above by b, so b is an upper bound for $[a,m)$. By LUB Axiom, the set has a Supremum.

Suppose $\sup\{m: f \text{ is negative on } [a,m)\}=c$.

This means $c<b$ because $f(b)>0$ and $f(x)<0$ on $[a,m)$.

$\therefore c<0$ or $c=0$

$c<0$ can't be true because if it is true, then there exists a number, $c\pm\varepsilon$, for which $f(c\pm\varepsilon)<0$ and then $\sup\{m: f \text{ is negative on } [a,m)\}=c\pm\varepsilon$, which contradicts the supposition. $\therefore f(c)=0$.

QED

2. Prove IVT

Proof:

Let $g(x)=f(x)-k$, for any $f(a)<k<f(b)$

Then $g(a)=f(a)-k<0$ and $g(b)=f(b)-k>0$

From the proof above we know there exists an $x=c$ such that $g(c)=0$.

$\therefore g(c)=f(c)-k$

$=0$

$\therefore f(c)=k$

QED