Week 5 and 6 Notes

Contunity

If a function is continuous, it must pass these 3 rules

1. c is in the domain of $f(x)$
2. $f(c)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Eg. Write a delta-epsilon proof to show that $f(x) = 2x+1$ is continuous at $x=5$.

1. Dom of $f(x) = (-\infty, \infty)$

   $5 \in$ Dom $f(x)$

2. $f(5) = 2(5)+1$

   $= 11$

3. Prove $\lim_{x \to 5} 2x+1 = 11$

$\forall \varepsilon > 0, \exists \delta > 0 \; |0 < |x-5| < \delta \rightarrow |2x-10| < \varepsilon$

$|2x-10| < \varepsilon$

$2|x-5| < \varepsilon$

$2\delta = \varepsilon$

$\delta = \varepsilon/2$

$|2x-10| < \varepsilon$

$2|x-5|$

$= 2\delta$

$= 2(\varepsilon/2)$

$= \varepsilon$

QED

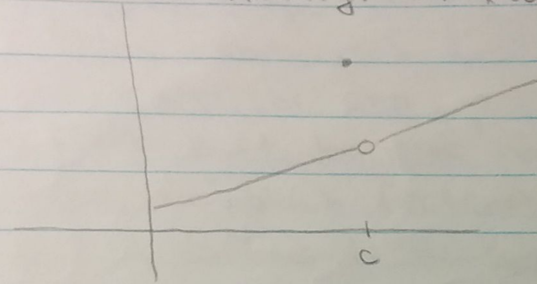
$\therefore \lim_{x \to 5} 2x+1 = 11$

$\therefore f(x)$ is continuous at $x=5$

Types of Discontinuities

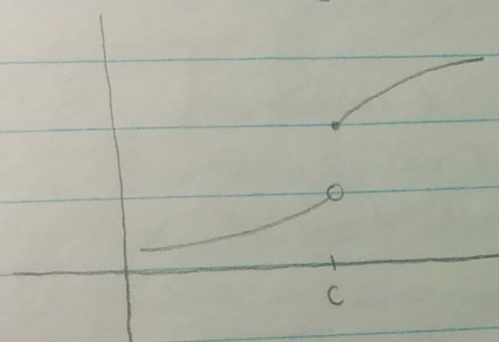
1. Removable Discontinuity: If $\lim_{x \to c} f(x)$ exists but isn't equal to $f(c)$

E.g.



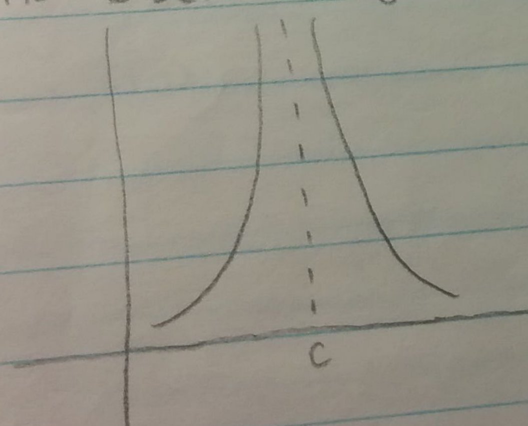
2. Jump Discontinuity: If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist but are not equal.

E.g.



3. Infinite Discontinuity: If one or both of $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ is infinite.

E.g.



Continunity at a point:
If function f is defined on open interval $(c-p, c+p)$, we say that f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

The precise definition of contunity at a point is:
If function f is defined on open interval $(c-p, c+p)$, we say that f is continuous at c if for each $\varepsilon > 0$, there exists $d > 0$ such that if $|x-c| < d$ then $|f(x) - f(c)| < \varepsilon$.

One Sided Limits

- $f(x)$ is cont from the right at number c if $\lim_{x \to c^+} f(x) = f(c)$.

$\forall \varepsilon > 0, \exists d > 0 \mid |x \in (c, c+d)| \rightarrow |f(x) - f(c)| < \varepsilon$

- $f(x)$ is cont from the left at number c if $\lim_{x \to c^-} f(x) = f(c)$

$\forall \varepsilon > 0, \exists d > 0 \mid |x \in (c-d, c)| \rightarrow |f(x) - f(c)| < \varepsilon$

Contunity on an Interval

- Function f is cont on open interval (a,b) if it is cont at every number in the interval

For all $p \in (a,b)$: $\lim_{x \to p} f(x) = f(p)$

- Function f is cont on closed interval $[a,b]$ if it is cont on open interval (a,b), right cont at a and left cont at b.

For all $p \in (a,b)$: $\lim_{x \to p} f(x) = f(p)$

$\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

3 Important Consequences of Contunity

1. Cont functions on closed interval [a,b] attain their min and max values on that interval.

2. Values of cont functions on closed interval go through every possible value between f(a) and f(b).

3. Cont functions on closed interval are bounded and attain their upper and lower bounds.

- Extreme Value Theorem:

If $f(x)$ is cont on closed interval $[a,b]$, then there exists some values M and m in $[a,b]$ such that $f(M)$ is the max value and $f(m)$ is the min value.

Proof:

Lemma

If f is cont on $[a,b]$, then f is bounded on $[a,b]$.

Proof:

Consider $\{x : x \in [a,b]$ and f is bounded on $[a,x]\}$

It is easy to see that this set is non-empty and bounded above by b. Thus, we can set $c = \text{lub}\{x : f$ is bounded on $[a,x]\}$

To argue $c=b$, suppose $c<b$. Since f is cont at c, it is bounded on $[c-\varepsilon, c+\varepsilon]$ for some $\varepsilon>0$. This means that f is bounded on $[a, c-\varepsilon]$ and on $[c-\varepsilon, c+\varepsilon]$, which means f is bounded on $[a, c+\varepsilon]$. However, this contradicts our choice of c, so $c=b$. This tells us that f is bounded on $[a,x]$, $x<b$. From the continuity of f, we know it is bounded on $[b-\varepsilon, b]$, $b-\varepsilon<b$. $\therefore$ f is bounded on $[a, b-\varepsilon]$ and $[b-\varepsilon, b]$ which means f is bounded on $[a,b]$.

QED

Extreme Value Theorem

By the lemma, f is bounded on $[a,b]$.

Set $M = \text{lub}\{f(x) : x \in [a,b]\}$

We have to show that there exists a c in $[a,b]$ such that $f(c)=M$

To do this, we set $g(x) = \dfrac{1}{M-f(x)}$

If $f(x) \neq M$, then $g(x)$ is cont on $[a,b]$ and bounded on $[a,b]$.

However, $g(x)$ cannot be bounded on $[a,b]$, which makes $f(x)=M$.

QED

Side Note *

The proof for m is similar

Intermediate Value Theorem:

If $f(x)$ is cont on $[a,b]$, then for any k strictly between $f(a)$ and $f(b)$, there exists at least one $c \in (a,b)$ such that $f(c) = k$

Proof:

Lemma:

If $f(x)$ is cont on $[a,b]$ and $f(a) < 0 < f(b)$, then $\exists\, c \in (a,b)$ for which $f(c) = a$

Proof

Let $f(a) < 0 < f(b)$

Since $f(a) < 0$ and $f(x)$ is cont on $[a,b]$, we know a number E such that $f(x)$ is negative on $[a, E)$. Since $[a, E)$ is bounded above by b, b is an upper bound for $[a, E)$. By the LUB Axiom, this set has a Sup. Let's suppose $\sup\{E : f \text{ is negative on } [a, E)\} = c$.

We know that $c \leq b$, but since $f(x)$ is negative on $[a, E)$ and $f(b) > 0$, $c \neq b$. Since $c < b$, $f(c) < 0$ or $f(c) = 0$. However, if $f(c) < 0$, then there exists a number $c \pm \varepsilon$ for which $f(c \pm \varepsilon) < 0$ and $c \pm \varepsilon$ becomes the new Sup. $\therefore f(c) = 0$

QED

Proof of IVT:

Let $g(x) = f(x) - k$ for any $f(a) < k < f(b)$

Then, $g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0$

From the lemma, we know $\exists\, x = c$, such that $g(c) = 0$.

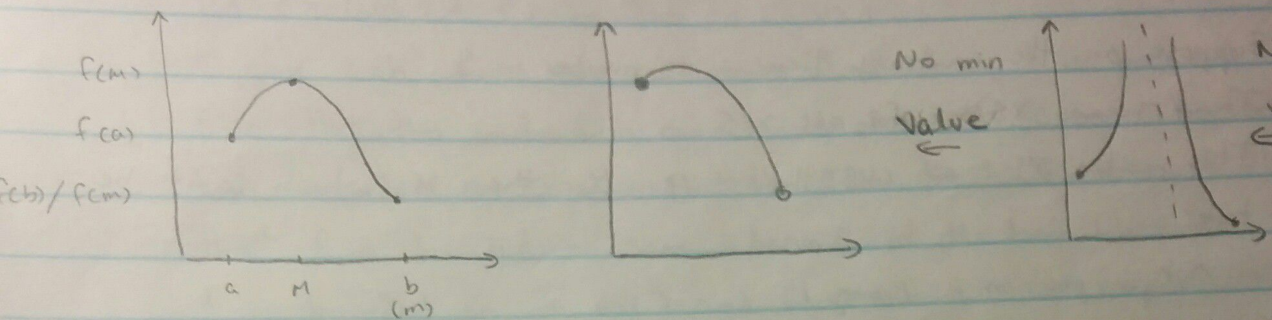
$\therefore\ g(c) = f(c) - k$

$0 = f(c) - k$

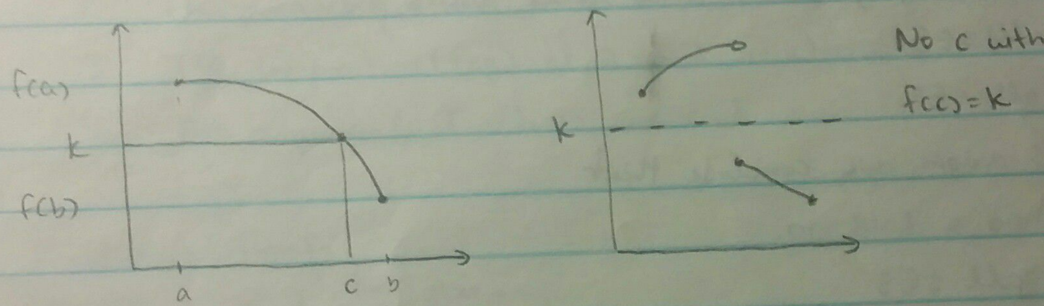
$f(c) = k$

QED

- Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then there exists values M and m in the interval $[a,b]$ such that $f(M)$ is the max value and $f(m)$ is the min value.



- Intermediate Value Theorem: If f is cont. on a closed interval $[a,b]$, then for any k between $f(a)$ and $f(b)$, there exists a $c \in (a,b)$ such that $f(c)=k$.



- A function can only change signs at a point $x=c$ only if $f(x)=0$, undefined or discontinuous at $x=c$.

- Least Upper Bound Axiom/ Sup:
Every nonempty set of real numbers that has an upper bound has at a LUB. The LUB may not directly be in the set, but can be found from the pattern.

E.g. $S=\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots\}$

The LUB of S is 1, because the numbers are approaching 1.

The largest element of a set, max S, exists if LUB S exists and LUB $S \in S$.

- If M is the LUB of set S and ε is a positive number, then there exists at least 1 number s in S such that $M-\varepsilon < s \leq M$.

Proof: Let $\varepsilon > 0$

$M-\varepsilon < s$

Suppose on the contrary there is no number in S.

That means $x \leq M-\varepsilon$ for all $x \in S$.

This makes $M-\varepsilon$ an LUB that is smaller than M, which can't be true.

$\therefore$ Our supposition is wrong

QED

- Every nonempty set of real numbers that has a lower bound has a GLB or Inf.

Proof: Suppose S in nonempty and has a lower bound k.

$k \leq s$, for all $s \in S$

$-s \leq -k$, for all $s \in S$

From the LUB axiom, we conclude that

$\{-s : s \in S\}$ has a LUB, m.

$-s \leq m$ for all $s \in S$

$-m \leq s$ for all $s \in S$

Suppose $-m < m_1 \leq s$, for all $s \in S$.

$-s \leq -m_1 < m$ for all $s \in S$

Thus m would not be LUB of $\{-s : s \in S\}$

- If m is the GLB of set S and ε is a positive number, then there exists a number s in S such that $m \leq s < m+\varepsilon$

- Bound Value Theorem:

If f is cont. on $[a,b]$, then f is bounded on $[a,b]$.

Proof: Consider $\{x : x \in [a,b]$ and f is bounded on $[a,x]\}$

Set $c = LUB\{x : f$ is bounded on $[a,x]\}$

Suppose $c<b$. From the continunity of f at c, f is bounded on $[c-\varepsilon, c+\varepsilon]$, $\varepsilon>0$. But this contradicts our choices of c. So $c=b$. This tells us f is bounded on $[a,x]$ with $x<b$. From the continunity of f, we know that f is bounded on some interval of the form $[b-\varepsilon, b]$. Since $b-\varepsilon<b$, f is bounded on $[a, b-\varepsilon]$ and $[b-\varepsilon, b]$. $\therefore$ f is bounded on $[a,b]$

- Properties of Limits

1. The sum/difference of 2 cont. functions is cont.
2. The product/quotient of 2 cont. functions is cont.
3. The composition $(f \circ g)$ of 2 cont functions is cont.
4. All polynomial functions are cont.

Limit Proofs

Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

1. Constant Multiple Law

Prove $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$

$$\lim_{x\to c} kf(x)$$

$$= \lim_{x\to c} k \cdot \lim_{x\to c} f(x)$$

$$= k \lim_{x\to c} f(x)$$

2. Addition Law

Prove $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

$\lim_{x \to c} f(x) = L$ means $\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0 < |x-c| < d_1 \Rightarrow |f(x) - L| < \varepsilon_1$

Let ε_1 be $\frac{\varepsilon}{2}$

$\lim_{x \to c} g(x) = M$ means $\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0 < |x-c| < d_2 \rightarrow |f(x) - M| < \varepsilon_2$

Let ε_2 be $\frac{\varepsilon}{2}$

Choose d to be $\min(d_1, d_2)$

Then, both $f(x)$ and $g(x)$ are within $\frac{\varepsilon}{2}$ of M and L.

$$-\frac{\varepsilon}{2} < f(x) - L < \frac{\varepsilon}{2}$$

$$+ \quad -\frac{\varepsilon}{2} < g(x) - M < \frac{\varepsilon}{2}$$

$$-\varepsilon < f(x) + g(x) - (L+M) < \frac{\varepsilon}{2}$$

QED

3. Product Rule ✳ Optional

Prove $\lim_{x \to c} [f(x) \cdot g(x)] = LM$

$\lim_{x \to c} [f(x) - L] = 0$ and $\lim_{x \to c} [g(x) - L] = 0$

Proof: $\lim_{x \to c} [f(x) - L]$

$= \lim_{x \to c} f(x) - \lim_{x \to c} L$

$= L - L$

$= 0$

The same logic applies to $\lim_{x \to c} [g(x) - M] = 0$.

Now, we have to prove $\lim_{x \to c} [f(x)-L][g(x)-M]=0$

$\lim_{x \to c} [f(x) - L] = 0$ means:

$\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0 < |x-c| < d_1 \to |f(x)-L| < \varepsilon_1$

$\lim_{x \to c} [g(x)-M] = 0$ means:

$\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0 < |x-c| < d_2 \to |g(x)-M| < \varepsilon_2$

Let $\varepsilon_1 = \sqrt{\varepsilon}$ and $\varepsilon_2 = \sqrt{\varepsilon}$

Choose d to be $\min(d_1, d_2)$

Then: $|f(x)-L| < \sqrt{\varepsilon}$ and $|g(x)-M| < \sqrt{\varepsilon}$

Multiplying the two together, we get $|(f(x)-L)(g(x)-M)| < \varepsilon$

$\therefore \lim_{x \to c} [f(x)-L][g(x)-M] = 0$

If we multiply $[f(x)-L][g(x)-M]$, we get $f(x)g(x) - f(x)(M) - g(x)(L) + LM$

$f(x)g(x) = [f(x)-L][g(x)-M] + f(x)(M) + g(x)(L) - LM$

$$\lim_{x \to c} f(x)g(x) = \lim_{x \to c} \left[[f(x)-L][g(x)-M] + f(x)M + g(x)L - LM \right]$$

$$= \lim_{x \to c} [f(x)-L][g(x)-M] + \lim_{x \to c} f(x)M + \lim_{x \to c} g(x)L - \lim_{x \to c} LM$$

$$= 0 + LM + LM - LM$$

$$= LM$$

QED

4. Difference Law

Prove $\lim\limits_{x \to c} [f(x) - g(x)] = L - M$

$$\lim_{x \to c} [f(x) - g(x)]$$

$$= \lim_{x \to c} [f(x) + (-g(x))]$$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} (-g(x))$$

$$= \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

$$= L - M$$

QED

5. Reciprocal Law

Prove $\lim\limits_{x \to c} \frac{1}{g(x)} = \frac{1}{M}, \quad M \neq 0$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x - c| < d \rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right|$$

$$= \left| \frac{M - g(x)}{M(g(x))} \right| \qquad (|M - g(x)| = |g(x) - M|) \text{ Side Note}$$

$$= \left| \frac{1}{M} \right| \left| \frac{1}{g(x)} \right| |g(x) - M|$$

Choose d_1 such that $|g(x) - M| < \frac{|M|}{2} \rightarrow |g(x)| > \frac{|M|}{2} \rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|}$

Choose d_2 such that $|g(x) - M| < \frac{M^2 \varepsilon}{2}$

Choose d to be $\min(d_1, d_2)$

$$\left| \frac{1}{M} \right| \left| \frac{1}{g(x)} \right| |g(x) - M| < \frac{2}{M^2} \left(\frac{M^2}{2} \right) (\varepsilon)$$

$$< \varepsilon$$

QED

6. Quotient Law * Optional

Prove $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

We can rewrite $\lim_{x \to c} \frac{f(x)}{g(x)}$ as $\lim_{x \to c} f(x)\left(\frac{1}{g(x)}\right)$

$\lim_{x \to c} f(x)\left(\frac{1}{g(x)}\right)$

$= \lim_{x \to c} f(x) \cdot \boxed{\lim_{x \to c} \frac{1}{g(x)}}$ ← We proved this in the reciprocal Law

$= L \cdot \frac{1}{M}$

$= \frac{L}{M}$ QED

7. Continunity of power function

Prove that $f(x) = x^k$ is continuous on $(-\infty, \infty)$

We have to prove $f(x)$ is cont on $(-\infty, \infty)$ first

1. Dom $f(x) = (-\infty, \infty)$
2. $f(c) = c^k$
3. $\lim_{x \to c} x^k = f(c) = c^k$

By the product rule, $\lim_{x \to c} x^k = \lim_{x \to c} x \cdot \lim_{x \to c} x \ldots$

$= c \cdot c \cdot c \ldots$

$= c^k$

QED

* Side Note *

We can also prove the continunity of $f(x) = x^{-k}$ and $f(x) = x^{p/q}$ using the same strategy.

9. Squeeze Theorem

Let $p>0$. Suppose that $\forall x$ such that $0<|x-c|<p$, $h(x)\le f(x)\le g(x)$.
If $\lim_{x\to c} h(x) = L$ and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(x) = L$

Proof: $\forall \varepsilon>0, \exists d_1>0 \mid 0<|x-c|<d_1 \to |h(x)-L|<\varepsilon$

$\forall \varepsilon>0, \exists d_2>0 \mid 0<|x-c|<d_2 \to |g(x)-L|<\varepsilon$

Choose d to be $\min(d_1, d_2)$

Then, $|h(x)-L|<\varepsilon$ and $|g(x)-L|<\varepsilon$

$\downarrow \qquad\qquad \downarrow$

$-\varepsilon < h(x)-L < \varepsilon \qquad -\varepsilon < g(x)-L < \varepsilon$

$L-\varepsilon < h(x) < L+\varepsilon \qquad L-\varepsilon < g(x) < L+\varepsilon$

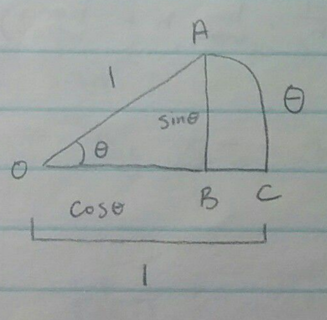
Since $L-\varepsilon \le h(x) \le f(x) \le g(x) \le L+\varepsilon$,

then $L-\varepsilon \le f(x) \le L+\varepsilon$

QED

10. Prove the continuity of $f(x) = \sin(x)$

1. Domain $\sin(x) = (-\infty, \infty)$
2. $f(c) = \sin(c)$
3. Find $\lim_{x\to c} \sin x = \sin c$



$\sin\theta = |AB|$

$0 \le |AB| \le \theta$

$\lim_{\theta\to 0} 0 = 0$ and $\lim_{\theta\to 0} \theta = 0$

$\therefore$ By S.T., $\lim_{\theta\to 0} \sin\theta = 0$

We also have to prove $\lim_{\theta\to 0} \cos\theta = 1$

$0 \le |BC| \le \theta$

$0 \le |1-\cos\theta| \le \theta$

$-1 \le -\cos\theta \le \theta - 1$

$1 \ge \cos\theta \ge 1-\theta$

$\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0} 1-\theta = 1$

$\therefore$ By S.T., $\lim_{\theta\to 0} \cos\theta = 1$

Now, Sub $x = c+h$ in. If $x \to c$, then $h \to 0$

$$\lim_{x \to c} \sin x$$

$$= \lim_{h \to 0} \sin(c+h)$$

$$= \lim_{h \to 0} (\sin c \cosh + \sinh \cos c)$$

$$= \lim_{h \to 0} \sin c \cdot \cosh + \lim_{h \to 0} \sinh \cos c$$

$$= \lim_{h \to 0} \sin c \cdot \lim_{h \to 0} \cosh + \lim_{h \to 0} \sinh \cdot \lim_{h \to 0} \cos c$$

$$= \sin c$$

QED

11. Prove the continuity of $f(x) = e^x$

We define e to be the number that $(1+h)^{\frac{1}{h}}$ approaches as h approaches 0.

$$\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e \quad \text{and} \quad \lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = 1$$

1. Dom $e^x = (-\infty, \infty)$
2. $f(c) = e^c$
3. $\lim_{x \to c} e^x = e^c$

Sub $x = c+h$

$$\lim_{h \to 0} e^x = \lim_{h \to 0} e^{c+h}$$

$$= \lim_{h \to 0} e^c \cdot \lim_{h \to 0} e^h$$

$$= e^c \cdot \lim_{h \to 0} (e^h + 1 - 1)$$

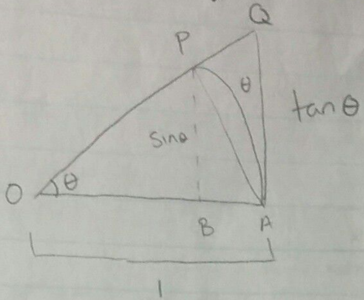
$$= e^c \cdot \lim_{h \to 0} \left(\frac{e^h - 1}{h} h + 1\right)$$

$$= e^c \left(\lim_{h \to 0} \frac{e^h - 1}{h} \cdot \lim_{h \to 0} h + \lim_{h \to 0} 1\right)$$

$$= e^c$$

QED

12 Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$



Area $\triangle OPA = \frac{\sin\theta}{2}$

Area Sector OPA $= \frac{\theta}{2}$

Area of OPA $\leq$ Area of Sector OPA $\leq$ Area OQA

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan\theta}{2}$$

$$\sin\theta \leq \theta \leq \tan\theta$$

$$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

$$1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$$

$\lim_{x \to 0} 1 = 1$ and $\lim_{x \to 0} \theta = 1$

$\therefore$ By S.T., $\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$

QED

13 Prove $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

$$\lim_{x \to 0} \frac{1-\cos x}{x} \left(\frac{1+\cos x}{1+\cos x} \right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1+\cos x}$$

$= 0$

QED

14 Prove if $\lim_{x \to c} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{0^+}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \infty$

$\lim_{x \to c} f(x) = 1$ means

$\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0 < |x-c| < d_1 \rightarrow |f(x) - 1| < \varepsilon_1$

$\downarrow$

$1 - \varepsilon_1 < f(x) < 1 + \varepsilon_1$

$\lim_{x \to c} g(x) = 0$ means

$\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0 < |x-c| < d_2 \rightarrow |g(x)| < \varepsilon_2$

$\downarrow$

$-\varepsilon_2 < g(x) < \varepsilon_2$

Choose d to be $\min(d_1, d_2)$.

That means $\frac{f(x)}{g(x)} > \frac{1-\varepsilon_1}{\varepsilon_2}$

Let ε_1 be $\frac{1}{2}$

Let ε_2 be $\frac{1}{2M}$

$$\frac{f(x)}{g(x)} > \frac{1/2}{1/2M}$$

$$> M$$

QED

15. If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is in the form of $\frac{1}{0^-}$, then $\frac{f(x)}{g(x)} = -\infty$

$\lim_{x\to c} f(x) = 1$ means

$\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0 < |x-c| < d_1 \Rightarrow 1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to c} g(x) = 0$ means

$\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0 < |x-c| < d_2 \Rightarrow -\varepsilon_2 < g(x) < \varepsilon_2$

Choose d to be $\min(d_1, d_2)$.

Then $\frac{f(x)}{g(x)} > \frac{1-\varepsilon_1}{\varepsilon_2}$

Let ε_1 be $\frac{1}{2}$.

Let ε_2 be $\frac{1}{2M}$

$\frac{f(x)}{g(x)} > M$

QED

16 If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

$\lim_{x\to\infty} f(x) = 1$ means:

$\forall \varepsilon_1 > 0, \exists N_1 > 0 \mid x > N_1 \Rightarrow 1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to\infty} g(x) = \infty$ means:

$\forall M > 0, \exists N_2 > 0 \mid x > N_2 \Rightarrow g(x) > M$

Choose N to be $\max(N_1, N_2)$

That means $\left|\frac{f(x)}{g(x)}\right| < \frac{1+\varepsilon_1}{M}$

Let $\varepsilon_1 = 1$ and let $M = \frac{2}{\varepsilon}$

$\left|\frac{f(x)}{g(x)}\right| = \varepsilon$

QED

17 If $\lim_{x\to-\infty} \frac{f(x)}{g(x)}$ is in the form of $\frac{1}{-\infty}$, then $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 0$

$\lim_{x\to-\infty} f(x) = 1$ means

$\forall \varepsilon_1 > 0, \exists N_1 < 0 \mid x < -N_1 \to 1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to-\infty} g(x) = -\infty$ means

$\forall M < 0, \exists N_2 < 0 \mid x < -N_2 \to g(x) < -M$

Choose N to be $\max(N_1, N_2)$

$\left|\frac{f(x)}{g(x)}\right| < \frac{1+\varepsilon_1}{M}$

Let $\varepsilon_1 = 1$ and let $\varepsilon_2 = -2/\varepsilon$

$\left|\frac{f(x)}{g(x)}\right| = M$

QED

18. Continunity of Composition Functions

If g is cont at c and f is cont at $g(c)$, then $f \circ g$ is cont. at c.

WTS: $\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \to |f(g(x)) - f(g(c))| < \varepsilon$

To show g is cont at c:

$\exists d > 0 \mid 0 < |x-c| < d \to |g(x) - g(c)| < d_1$ ①

To show f is cont at $g(c)$:

$\exists d_1 > 0 \mid 0 < |g(x) - g(c)| < d_1 \to |f(g(x)) - f(g(c))| < \varepsilon$ ②

Combining 1 and 2, we get

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \to |f(g(x)) - f(g(c))| < \varepsilon$

QED