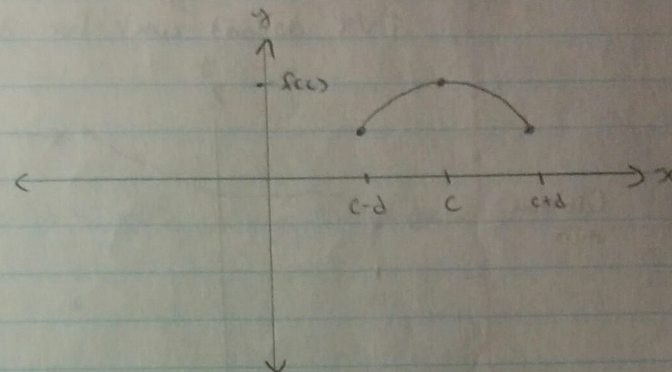
Week 10/11 Notes

1. Fermat's Theorem for Local Extrema:
If $f(x)$ has a local max at an interior point c and $f'(c)$ exists, then $f'(c)=0$.

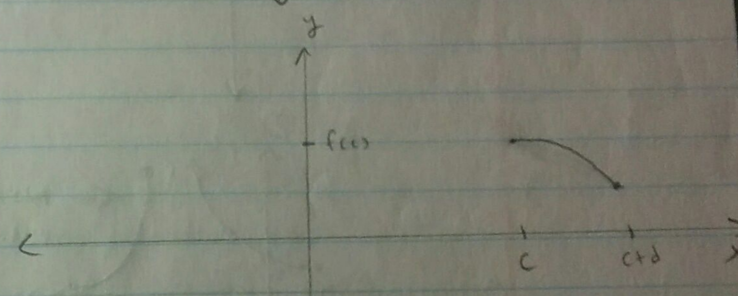
Proof:
From the definiton $f(x)$ has a local max at c if $f(x) \leq f(c)$ on $x \in (c-d, c+d)$.

$$f'_+(c) = \begin{cases} x \in (c, c+d) \\ f(x)-f(c) \leq 0 \\ x-c \geq 0 \end{cases}$$



$$f'_+(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$$

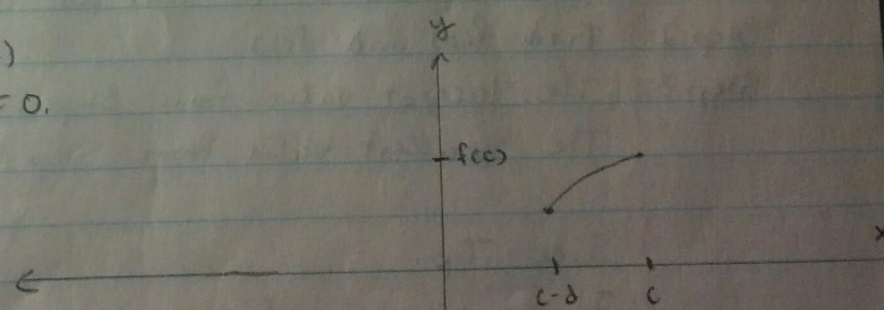
$$f'_-(c) = \begin{cases} x \in (c-d, c) \\ f(x)-f(c) \geq 0 \\ x-c \leq 0 \end{cases}$$



Since $x \in (c, c+d)$ then $x > c$.
However, since $f(c)$ is a local max, $f(x) \leq f(c)$

$$f'_-(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$$

However, because $f'(c)$ exists, $f'_+(c) = f'_-(c) = 0$.
QED



* Note *
The inverse is false.
$f'(c)=0$ doesn't mean $f(c)$ is a local max.
$f(c)$ can be a local min.

Since $x \in (c-d, c)$, $x \leq c$.
Since $f(c)$ is a local ma
$f(x) \leq f(c)$

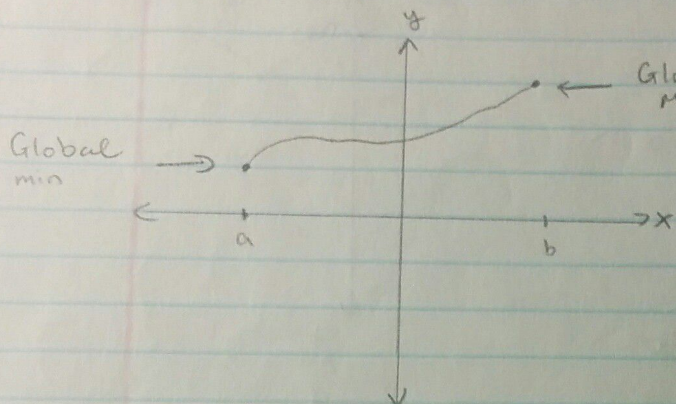
2 Extreme Value Theorem (EVT):

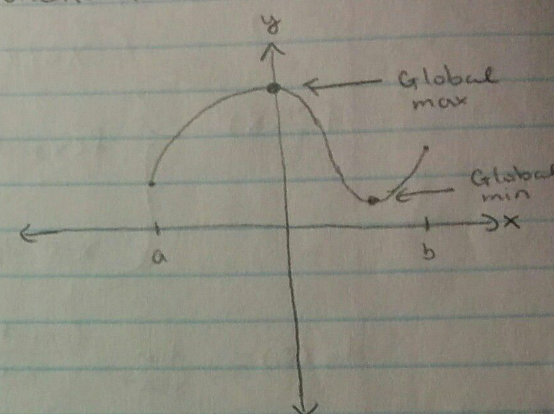
If $f(x)$ is cont on closed interval $[a,b]$, then $f(x)$ attains an absolute max or global max and an absolute min or global min at some numbers in $[a,b]$

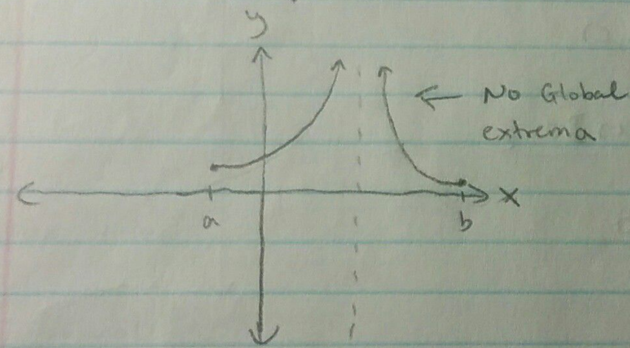
abs max = global max

abs min = global min

EVT doesn't work for discont functions.







Use the Closed Interval Method to find global extrema

Step 1: Find the values of $f(x)$ at the critical

Step 2: Find $f(a)$ and $f(b)$

Step 3: The largest value from Step 1 and 2 is the abs max

The smallest value from Step 1 and 2 is the abs min

Side Tip:

If one or more of the endpoints is open, eg. $(a,b]$ or $[a,b)$ or (a,b) instead of $[a,b]$, do step 1 the same way, but for step 2, you find the limit for all open intervals. If an open interval is greater or less than all critical points, an abs max or min won't exist.

E.g. Find the abs max and abs min of the function
$f(x) = x^3 - 27x + 1$ at $[-1, 6]$

Step 1:

Find the critical points

$f'(x) = 3x^2 - 27$

$0 = 3x^2 - 27$

$= x^2 - 9$

$9 = x^2$

$x = \pm 3$

We reject $x = -3$ because it's not within the interval $[-1, 6]$.

$f(3) = (3)^3 - 27(3) + 1$

$= 27 - 81 + 1$

$= -53$

Step 2:

Find the f(x) at the boundary points

$f(-1) = (-1)^3 - 27(-1) + 1$

$= -1 + 27 + 1$

$= 27$

$f(6) = (6)^3 - 27(6) + 1$

$= 216 - 162 + 1$

$= 55$

Step 3:

Abs max is at $x = 6$.

Abs min is at $x = 3$.

E.g. Find the abs max and abs min of $f(x)$ in the interval $[-1, 6)$

Step 1:

Look at previous page.

Step 2:

Since there is an open interval at $x=6$, we have to find the left sided limit at $x=6$.

$f(-1) = 27$

$$\lim_{x \to 6^-} x^3 - 27x + 1$$

$$= \lim_{x \to 6^-} (6)^3 - 27(6) + 1$$

$$= 55$$

Step 3

Abs min is at $x=3$

There is no abs max because $f(6)$ has the higlest value but its an open interval.

Side note:

When dealing with open intervals, you find the right limit for the smaller boundary and right limit for the larger boundary.

E.g. (a, b)

Find left hand limit

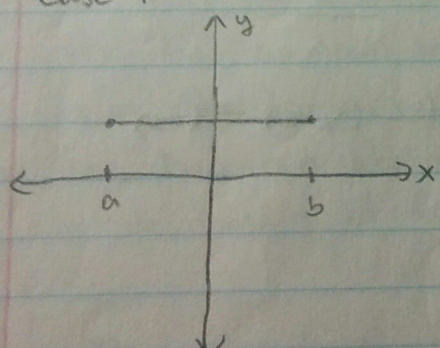
Find right hand limit

3. Rolle's Theorem:
If $f(x)$ is cont on $[a,b]$, diff on (a,b) and $f(a)=f(b)$, then there exists at least 1 number $c \in (a,b)$ such that $f'(c)=0$.
Proof.
If $f(x)$ is cont on $[a,b]$, diff on (a,b) and $f(a)=f(b)$, then by EVT, $f(x)$ has an abs max and an abs min between $[a,b]$.

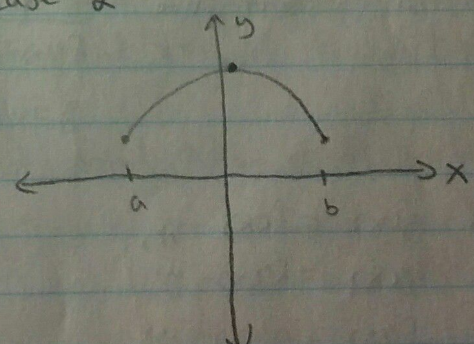
- Case 1:
$f(x) = c$, c is a constant
Since $f'(x)=c$, then $f'(x)=0$ for all numbers in $[a,b]$.
- Case 2:
$f(x) \neq c$, and $f(x)$ attains one of the global extremum on the endpoints of $[a,b]$. Since $f(a)=f(b)$, then $f(x)$ attains local extremum at point c on open interval (a,b) and by Fermat's Theorem, $f'(c)=0$.
- Case 3:
$f(x) \neq c$ and $f(x)$ attains both global extrema at the interior points of $[a,b]$ then $f(x)$ attains its local extrema in the open interval (a,b) and by Fermat's theorem $f'(c)=0$.

$\therefore$ Rolle's Theorem is proved for any function.
QED

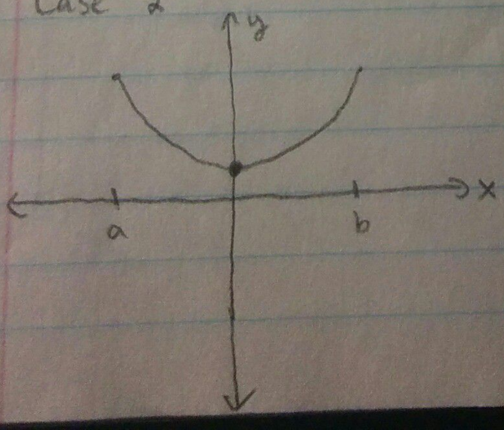
Case 1



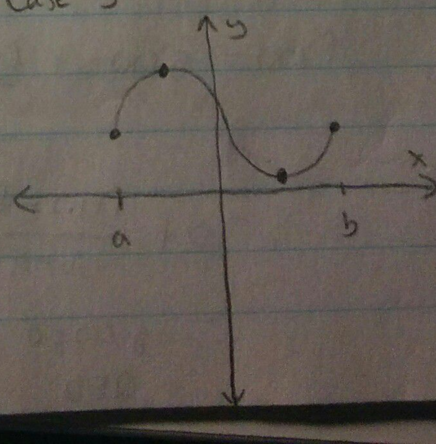
Case 2



Case 2

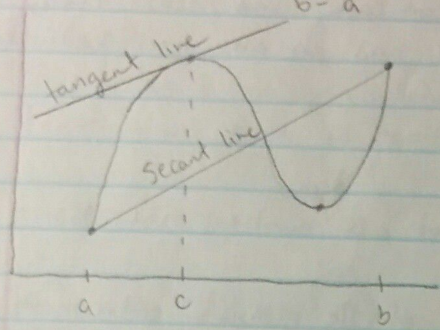


Case 3



4. Lagrange's Theorem / Mean Value Theorem (MVT):

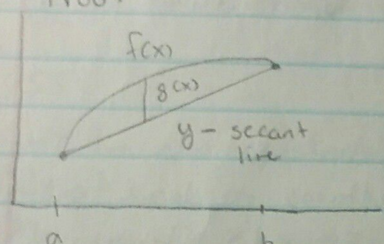
If $f(x)$ is cont on the closed interval $[a,b]$ and diff on open interval (a,b) then there exists at least 1 number $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.



$\frac{f(b)-f(a)}{b-a}$ is average rate of change on $[a,b]$.

$f'(c)$ is instanteous rate of change at $c \in (a,b)$.

Proof



let $g(x) = f(x) - y$

Since $g(x)$ is cont on $[a,b]$, $g(x)$ is diff on (a,b) and $g(a) = g(b)$, By Rolle's Theorem, $\exists\, c \in (a,b)$ such that $g'(c) = 0$

Eqn of secant line: $\frac{y - f(a)}{f(b)-f(a)} = \frac{x-a}{b-a}$

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Since $g(x) = f(x) - y$,

$g'(x) = f'(x) - y'$

$f'(x) = g'(x) + y'$

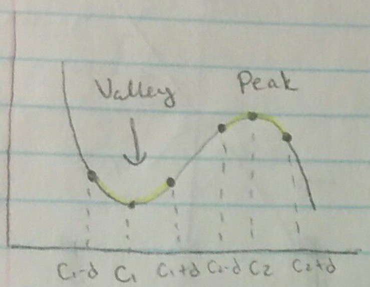
$$f'(x) = g'(x) + \frac{f(b)-f(a)}{b-a}$$

$$= 0 + \frac{f(b)-f(a)}{b-a}$$

$\therefore\ g'(c) = 0$

QED

5. Local (Relative) Extrema
Let $f(x)$ be a function defined on open interval (a,b).
$f(x)$ has a local min at $x=c$ if there exists some $d>0$ such that $f(c) \le f(x)$ for all $x \in (c-d, c+d)$.
$f(x)$ has a local max at point $x=c$ if there exists some some $d>0$ such that $f(c) \ge f(x)$ for all $x \in (c-d, c+d)$.

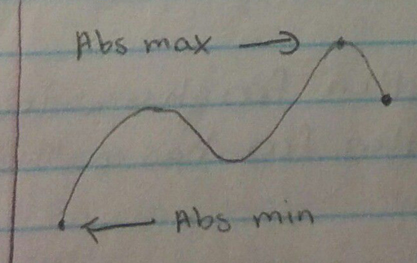


A local extrema can either be a peak or a valley.

To find local extrema, calculate $f'(x)=0$.

6. Critical Point:
If $x=c$ is in the domain of $f(x)$ such that either $f'(c)=0$ or $f'(c)=$ undefined, then that point is a critical point.
$f'(x)=0$ or $f'(x)=$ undefined will calculate critical point(s)

7. Global (Absolute) Extrema
Let $f(x)$ be a function defined on closed interval $[a,b]$.
$f(x)$ has an abs max at point $x=c$ if $f(c) \ge f(x)$ for all x in the domain of $f(x)$.
$f(x)$ has an abs min at point $x=c$ if $f(c) \le f(x)$ for all x in the domain of $f(x)$.



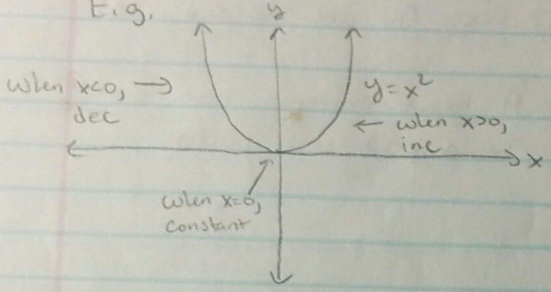
8. How derivatives affect the shape of a graph:

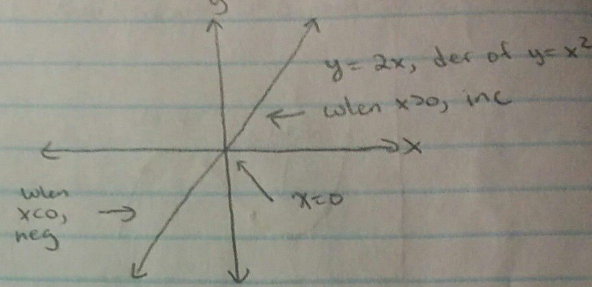
a) If $f'(x) > 0$ for all $x \in (a,b)$ then $f(x)$ is inc on (a,b)

b) If $f'(x) < 0$ for all $x \in (a,b)$ then $f(x)$ is dec on (a,b)

c) If $f'(x) = 0$ for all $x \in (a,b)$ then $f(x)$ is constant on (a,b)

E.g.





9. First Derivative Test

a) If $f'(x)$ changes from pos to neg at c, then $f(x)$ has a local max at c.

b) If $f'(x)$ changes from neg to pos at c, then $f(x)$ has a local min at c.

c) If $f'(x)$ doesn't change sign at c, then $f(x)$ has no max or min at c

10. Concavity Test

$f(x)$ concaves up if $f''(x) > 0$

$f(x)$ concaves down if $f''(x) < 0$

To find the inflection points, set $f''(x)$ to 0 and solve for x. An inflection point only exists iff $f''(x)$ changes its sign at the points you solved for x.

Second Derivative Test

a) If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ has a local max at c

b) If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ has a local min at c

II. Curve Sketching

E.g. Draw the graph of $f(x) = \frac{1}{x} + \frac{1}{x^2}$

Step 1. Find the Domain of $f(x)$

In this case, Dom $f(x)$: $x \in \mathbb{R} \mid x \neq 0$

Step 2. Find the x and y intercepts

To find x-int, sub $y=0$ and solve for x.

To find y-int, sub $x=0$ and solve for y.

In this case:

x-int:

$$0 = \frac{1}{x} + \frac{1}{x^2}$$

$$= \frac{x+1}{x^2}$$

$x = -1$

$(-1, 0)$

y-int:

There does not exist a y-int because x can't be 0.

DNE

Step 3. Find the symmetry of $f(x)$.

An even function exists if $f(-x) = f(x)$.

An odd function exists if $f(-x) = -f(x)$.

In this case, $f(-x) = \frac{1}{-x} + \frac{1}{x^2}$

$$f(x) = \frac{1}{x} + \frac{1}{x^2}$$

$$-f(x) = \frac{1}{x} - \frac{1}{x^2}$$

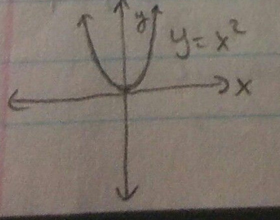
$\therefore$ Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, $f(x)$ is neither even nor odd.

Remainder *

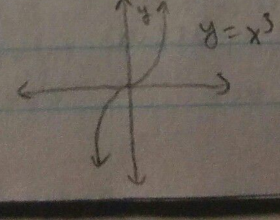
An even function has symmetry about the y-axis.

An odd function has symmetry about the origin.

Even Function



Odd Function



Step 4. Find asymptotes of $f(x)$.
There are 3 types of asymptotes, vertical (VA), horizontal (HA), and slant (SA).
Vertical Asymptotes:
To find VA, factor the denominator and see where $x = 0$.

E.g. $f(x) = \frac{1}{x} + \frac{1}{x^2}$

In this case, VA would be at $x = 0$.

Horizontal Asymptote:
To find HA compare the highest degree of the numerator with the highest degree of the denominator. If the highest degree of the numerator is less than the highest degree of the denominator, $y = 0$ is the HA. If they are equal, HA = $\frac{\text{leading coefficient of num}}{\text{leading coefficient of deno}}$

E.g. $f(x) = \frac{3x^2 + 5x - 4}{4x^3 - 3x^2 + 8}$

Since $2 < 3$, HA: $y = 0$

$f(x) = \frac{3x^4 + 5x - 4}{4x^4 + 5x - 8}$

Since $4 = 4$, HA: $y = \frac{3}{4}$

In this case, $f(x) = \frac{x+1}{x^2}$, so HA: $y = 0$

Slant Asymptote:
Only occurs if the leading degree of the numerator is 1 more than the leading degree of the denominator.

E.g. $f(x) = \frac{5x^4 - 3x + 8}{3x^3 - 4x + 3}$

Since $4 = 3 + 1$, a slant asymptote occurs.

To find the slant asymptote, you divide the numerator by the denominator.

E.g. $f(x) = \dfrac{5x^4 - 3x + 8}{3x^3 - 4x + 3}$

$$\begin{array}{r} \frac{5}{3}x \\ 3x^3-4x+3 \overline{\smash{)}\, 5x^4 - 3x + 8} \\ \underline{5x^4 \qquad\qquad} \\ -3x + 8 \end{array}$$

∴ The eqn of SA is $y = \frac{5}{3}x$

Side Notes *

A function can't cross VA but can cross HA and SA.
A function cannot have both HA and SA. It either has a HA or a SA or neither. To check if f(x) crosses HA or SA, Solve f(x) = HA or f(x) = SA. The value(s) of x solved will intersect HA or SA.

Step 5. Find Critical Points

To find the critical points, solve $f'(x) = 0$ or $f'(x) = DNE$. Then, to find if f(x) is increasing or decreasing,

In this case, $f'(x) = \dfrac{-1}{x^2} - \dfrac{2}{x^3}$

$$0 = \frac{-x-2}{x^3}$$

$$= -x - 2$$

$$= x + 2$$

$x = 0$ and $x = -2$ are the critical points

	$x < -2$	$-2 < x < 0$	$x > 0$
$-(x+2)$	+	−	−
x^3	−	−	+
Sign	−	+	−
	↑ f(x) is dec when $x < -2$	↑ f(x) is inc when $-2 < x < 0$	↑ f(x) is dec when $x > 0$

Step 6. Concavity

To find the inflection points, solve $f''(x)=0$ or $f''(x)=$ DNE.

To find where $f(x)$ is concaving up or down, draw a sign chart.

E.g. $f(x) = \frac{1}{x} + \frac{1}{x^2}$

$$f'(x) = \frac{-x-2}{x^3}$$

$$f''(x) = \frac{2x+6}{x^4}$$

Since x^4 is in the denominator, $x \neq 0$.

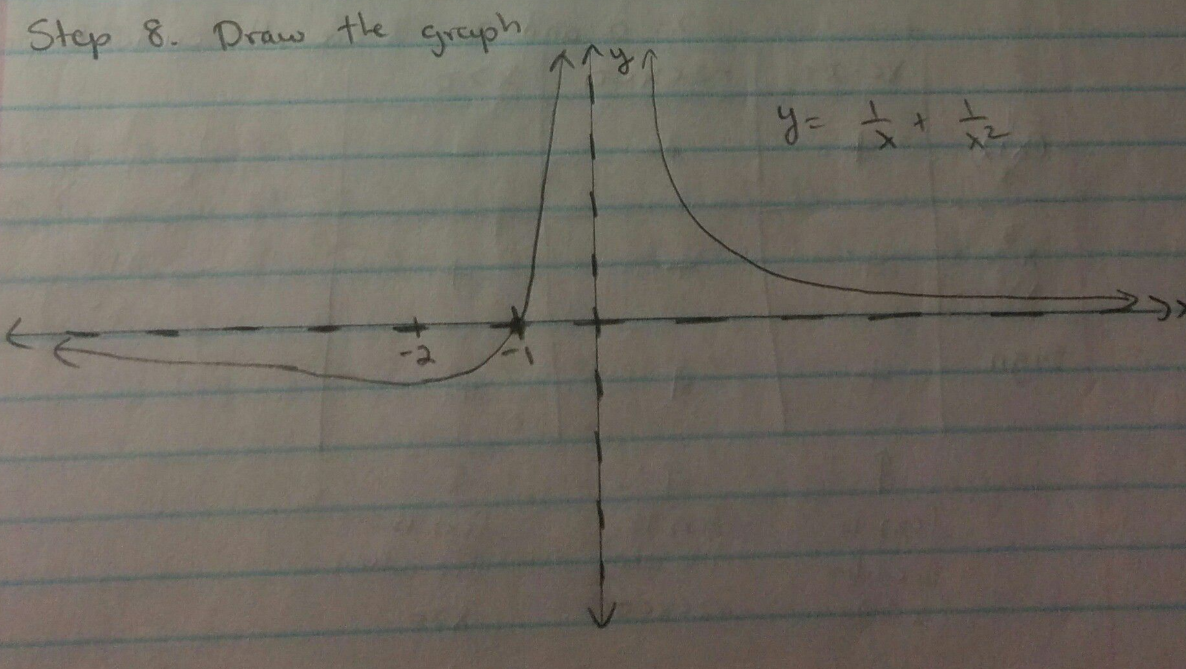
$0 = 2x+6$

$x = -3$

$\therefore$ $x=0$ and $x=-3$ are the inflection points

	$x<-3$	$-3<x<0$	$x>0$
$2x+6$	$-$	$+$	$+$
x^4	$+$	$+$	$+$
Sign	$-$	$+$	$+$
Concavity	$\cap$	$\cup$	$\cup$

Step 8. Draw the graph

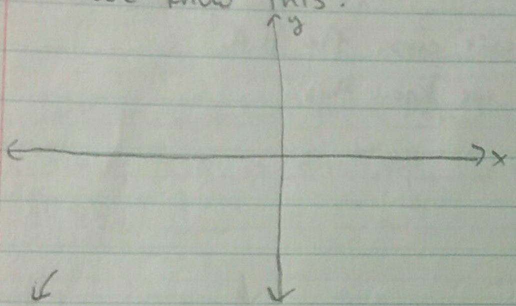


Side Note *

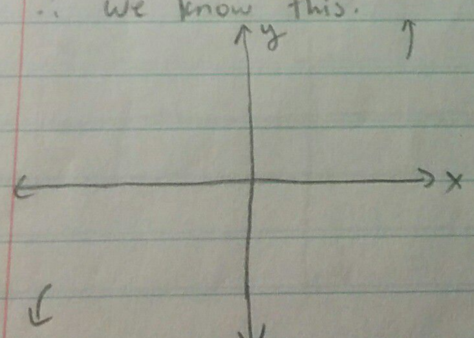
To see how your function behaves at the endpoints, sub in a large negative value for x and see if f(x) is positive or negative and a large positive value for x and see if f(x) is positive or negative.

E.g. Consider the function $f(x) = x^5 - x^4$.

1. Sub a large negative number for x.
Let's say $x = -10$
$f(-100) = -110{,}000$
$\therefore$ We know this:



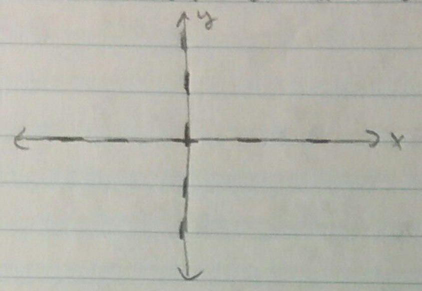
2. Sub in a large positive number for x.
Say $x = 10$
$f(100) = 90{,}000$
$\therefore$ we know this:



Suppose you have V.A., H.A. or S.A.

E.g. Consider the function $f(x) = \frac{x-1}{x}$

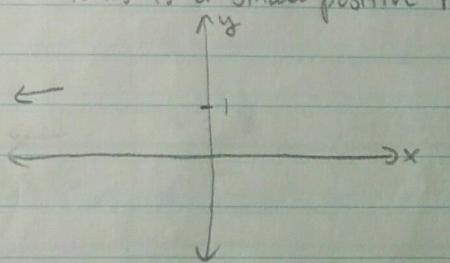
There is a V.A. at $x=0$ and a H.A. at $y=1$.



1. Sub in -10 for x

$f(-10) = 1.1 \rightarrow$ Since $f(x) > 1$, $f(x)$ doesn't cross the H.A.

Since $f(-10)$ is a small positive number, we know this:



2. Sub in a value for x that is close to $x =$ V.A., but smaller

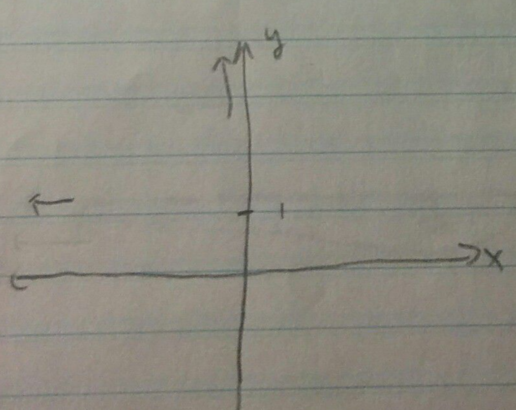
Since V.A. occurs when $x=0$, sub $x = -\frac{1}{2}$

$$f(-\tfrac{1}{2}) = \frac{-\frac{1}{2} - 1}{-\frac{1}{2}}$$

$$= -2(-\tfrac{3}{2})$$

$$= 6$$

Since $f(-\frac{1}{2})$ is positive, we know this:

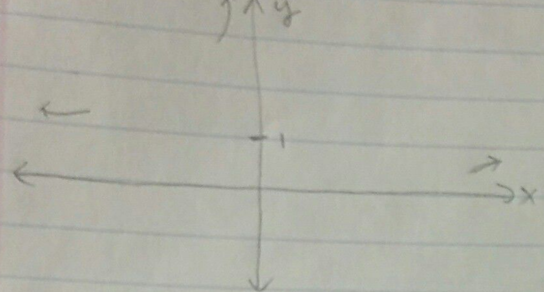


3. Sub in 10 for x

$$f(10) = \frac{10-1}{10}$$

$$= 0.9 \rightarrow$$ Since $f(x) < 1$, we know $f(x)$ doesn't cross the H.A.

∴ We know this:



4. Sub in a value for x that is close, but greater than the V.A.

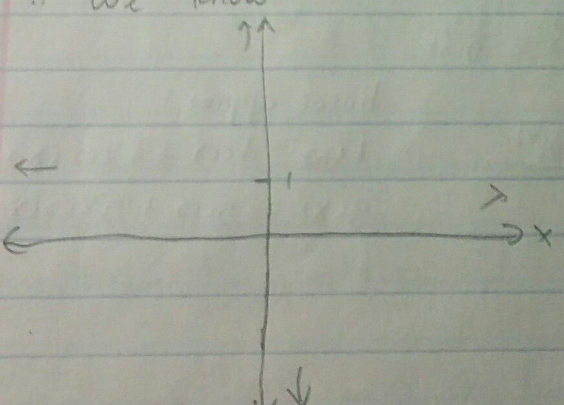
Since $x=0$ is the V.A, sub in $\frac{1}{2}$ for x

$$f(\tfrac{1}{2}) = \frac{\frac{1}{2}-1}{\frac{1}{2}}$$

$$= 2(-\tfrac{1}{2})$$

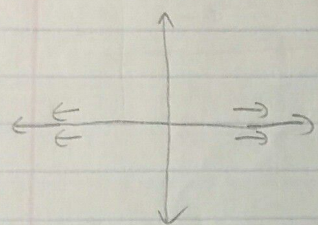
$$= -1$$

∴ We know

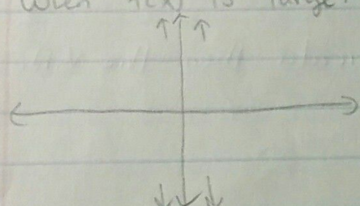


When you sub a large number for x, and f(x) is a small number, you know that f(x) will be approaching the H.A. Otherwise, if f(x) is a large number, then it will either go up or down.

When f(x) is small:



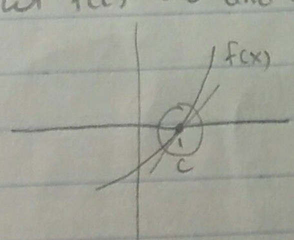
When f(x) is large:

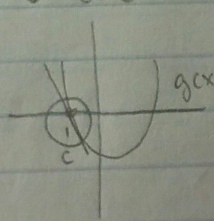


12. L'hopital's Rule

Proof:

Let $f(c) \to 0$ and $g(c) \to 0$ and $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$





linear approx:

$f(x) = f(c) + f'(c)(x-c)$

$g(x) = g(c) + g'(c)(x-c)$

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

$$= \lim_{x \to c} \frac{\overset{0}{\cancel{f(c)}} + f'(c)\cancel{(x-c)}}{\underset{0}{\cancel{g(c)}} + g'(c)\cancel{(x-c)}}$$

$$= \lim_{x \to c} \frac{f'(c)}{g'(c)}$$

L'hopital's Rule

If we have one of the following cases:

1. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$

2. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$

$\Big\rangle$ c can be a real number or $\pm\infty$

we will do $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ to solve it.

E.g. $\lim_{x \to 0} \frac{\sin x}{x}$

Subbing $x=0$ will give us $\frac{0}{0}$.

$\therefore$ Using l'hopital's rule, we get $\lim_{x \to 0} \frac{\cos x}{1}$

$= 1.$

E.g. $\lim_{x \to \infty} \frac{e^x}{x^2}$

Subbing $x=0$ will give us $\frac{\infty}{\infty}$.

$\therefore$ Using l'hopital's rule, we get $\lim_{x \to \infty} \frac{e^x}{2x}$.

This still gives us $\frac{\infty}{\infty}$, so apply l'hopital's rule again.

This time, we get $\lim_{x \to \infty} \frac{e^x}{2}$

$= \infty$

L'hopital's rule can be used more than once.

E.g. $\lim_{x \to 0^+} x\ln(x)$

This is in the form of $0(\infty)$, which is still indeterminate, but can't directly be solved with l'hopital's rule.

To fix this, we do $\lim_{x \to 0^+} \frac{\ln x}{1/x}$. Now, we can

apply l'hopital's rule. $\lim_{x \to 0^+} \frac{\ln x}{1/x}$

$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$

$= \lim_{x \to 0^+} -x = 0$

To solve problems that have $(0)(\pm\infty)$, turn $f(x)\cdot g(x)$ into $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$.

If you turned the product into a fraction and find that you're making no progress, turn it the other way.

Lastly, if $f(x)$ is in any of these forms:

1. 1^∞
2. 0^0
3. ∞^0

take the ln of both sides and solve it.

E.g. $\lim_{x\to\infty} x^{\frac{1}{x}}$

This is in the form of ∞^0, so we need to take the ln of both sides.

$y = x^{\frac{1}{x}}$

$\ln(y) = \frac{1}{x}\ln(x)$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

However, in the beginning, I said $y = x^{\frac{1}{x}}$, not $\ln(y)$.

But, remember $e^{\ln y} = y$.

$$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln y}$$

$$= e^0$$

$$= 1$$

$$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = 1$$

1. If $\lim_{x\to c} \ln(f(x)) = L$, then $\lim_{x\to c} f(x) = e^L$

2. If $\lim_{x\to c} \ln(f(x)) = \infty$, then $\lim_{x\to c} f(x) = \infty$

3. If $\lim_{x\to c} \ln(f(x)) = -\infty$, then $\lim_{x\to c} f(x) = 0$

This is because $\lim_{x\to c} \ln(f(x)) = \ln\left(\lim_{x\to c} f(x)\right)$

1. $\lim_{x\to c} \ln(f(x)) = L$

$\ln\left(\lim_{x\to c} f(x)\right) = L$

$e^{\ln\left(\lim_{x\to c} f(x)\right)} = e^L$

$\lim_{x\to c} f(x) = e^L$

2. $\lim_{x\to c} \ln(f(x)) = \infty$

$\ln\left(\lim_{x\to c} f(x)\right) = \infty$

$e^{\ln\left(\lim_{x\to c} f(x)\right)} = e^{\infty}$

$\lim_{x\to c} f(x) = e^{\infty}$

$= \infty$

3. $\lim_{x \to c} \ln(f(x)) = -\infty$

$$\ln\left(\lim_{x \to c} f(x)\right) = -\infty$$

$$e^{\ln\left(\lim_{x \to c} f(x)\right)} = e^{-\infty}$$

$$\lim_{x \to c} f(x) = e^{-\infty}$$

$$= 0$$